Online appendix to Overlapping Ownership, R&D Spillovers and Antitrust Policy

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Contents

In Appendix A we provide some technical details and proofs of results in the paper in Section A.1, and analysis and results for the three model specifications in Section A.2. In Appendix B we develop the Bertrand model with product differentiation.

A Proofs and the three model specifications

A.1 General model: proofs

A.1.1 Overlapping ownership and λ

Common ownership Consider an industry with n firms and $I \geq n$ investors; we let i and j index (respectively) investors and firms. The share of firm j owned by investor i is υ_{ij}, and the parameter ζ_{ij} captures the extent of i's control over firm j. The total (portfolio) profit of investor i is $\pi^i = \sum_k \upsilon_{ik}\pi_k$, where π_k are the profits of portfolio firm k. The manager of firm j takes into account shareholders' incentives (through the control weights ζ_{ij}) and maximizes a weighted average of the shareholders' portfolio profits:

$$\sum_{i=1}^{I} \zeta_{ij}\pi^i = \left(\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ij}\right)\pi_j + \sum_{i=1}^{I} \zeta_{ij} \sum_{k\neq j}^{n} \upsilon_{ik}\pi_k.$$

It is immediate dividing by $\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ij}$ that the objective of the manager can be rewritten as

$$\phi_j = \pi_j + \sum_{k\neq j}^{n} \lambda_{jk}\pi_k, \text{ where } \lambda_{jk} \equiv \frac{\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ik}}{\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ij}}.$$

The parameter λ_{jk} is the relative weight that the manager of firm j places on the profit of firm k in relation to the own profit (of firm j) and reflects the control of firm j by investors with financial interests in firms j and k. For the manager of firm j to put weight on the interest of investor i we need $\zeta_{ij}\upsilon_{ik} > 0$: investor i has to have a stake in firm k ($\upsilon_{ik} > 0$) *and* some control over firm j ($\zeta_{ij} > 0$). The weight λ_{jk} is larger the more firm j is controlled (high ζ_{ij}) by investors with high stakes in firm k (high υ_{ik}) and the less concentrated the ownership and control of firm j (low denominator $\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ij}$). The numerator $\sum_{i=1}^{I} \zeta_{ij}\upsilon_{ik}$ is a measure of the ownership concentration and control of firm k. As the ratio λ_{jk} increases, the influence of the common owners of firm k over the manager of firm j increases.

We next discuss the cases of silent financial interests and proportional control. In both cases we assume that each firm has a reference shareholder and each investor acquires a share α of the firms which are not under his control. The reference shareholder keeps an interest $1-(I-1)\alpha$ in his firm and we assume that $\alpha I < 1$ so that $1-(I-1)\alpha > \alpha$.

Silent Financial Interest (SFI). In this case, each owner (i.e., the majority or dominant shareholder) i retains full control of the acquiring firm and is entitled to a share α of the acquired firms' profits—but exerts no influence over the latter's decisions. Then $\lambda^{\text{SFI}} = \alpha/\left[1-(I-1)\alpha\right]$ is just the ratio of the share on an acquired firm k ($\upsilon_{ik} = \alpha$ in k, numerator of λ_{jk}) over the share in the own firm j ($\zeta_{ij}\upsilon_{ij} = 1-(I-1)\alpha$, denominator of λ_{jk}).[1] The result is that λ_{jk} is increasing in the number of investors I since when I increases investor i has less of a financial interest in his own firm (and when α increases then on a double account λ_{jk} increases). The driving force is that λ_{jk} increases as the size of the interest of undiversified shareholders diminishes. The upper bound of common-ownership is $\alpha = 1/I$, in which case $\lambda^{\mathrm{SFI}} = 1$.

Proportional Control (PC). Under proportional control, the firm's manager accounts for shareholders' own-firm interests in *other* firms in proportion to their respective stakes $\zeta_{ij} = \upsilon_{ij}$. In this case we have that $\lambda_{jk} = \left(\sum_{i=1}^{I} \upsilon_{ij}\upsilon_{ik}\right) / \left(\sum_{i=1}^{I} {\upsilon_{ij}}^2\right)$, where the denominator is the HHI on ownership shares of firm j and under symmetry λ^{PC} equals

$$\left\{2[1-(I-1)\alpha]\alpha + (I-2)\alpha^2\right\} / \left\{[1-(I-1)\alpha]^2 + (I-1)\alpha^2\right\}.^2$$

As with SFI, here $\lambda^{\mathrm{PC}} = 1$ when $\alpha = 1/I$. For $\alpha < 1/I$, then λ^{PC} is increasing in both I and α. The effects are more complex with proportional control but the relative weight of the profit of k over j ends up being monotone in the number of investors I and α. Both the numerator and denominator of λ^{PC} decrease with I but the denominator decreases more indicating that the ownership concentration of the firm's manager decreases by more than the one of other firms when I increases, inducing the manager to put a lower weight on the profits of other firms. The driving force again is the decline in the interest of the undiversified stake of reference investors $1-(I-1)\alpha$ as I or α increase.

Cross-ownership We assume here that each of the n firms may acquire their rivals' stock in the form of investments with no control rights. The profit of firm j is given by $\phi_j = \pi_j + \sum_{k\neq j} \alpha_{jk}\phi_k$, where α_{jk} is the firm j's ownership stake in firm k. One can derive the profit for each firm by denoting $\boldsymbol{\phi} \equiv (\phi_1, ..., \phi_n)'$ and $\boldsymbol{\pi} \equiv (\pi_1, ..., \pi_n)'$, and solving the matrix equation: $\boldsymbol{\phi} = \boldsymbol{\pi} + \mathbf{A}\boldsymbol{\phi}$, where $\mathbf{A}$ is the $n \times n$ matrix with the ownership stakes with 0's in the diagonal and α_{jk} off-diagonal. Thus, $\boldsymbol{\phi} = \boldsymbol{\Theta}\boldsymbol{\pi}$, where $\boldsymbol{\Theta} = (\mathbf{I}-\mathbf{A})^{-1}$ is the inverse of the Leontief matrix; its coefficients θ_{jk} represent the effective or imputed stake in firm k's profits received by a "real" equity holder with a 1% direct stake in firm j. We examine the symmetric case: $\alpha_{jk} = \alpha_{kj} \equiv \alpha$ for all $j \neq k$, and $\alpha_{jj} = 0$ for all j. The formula for the coefficients of matrix Θ when stakes are

[1] If i owns and controls j, then (i) $\zeta_{ij} = 1$ and $\zeta_{ik} = 0$ for $k \neq j$; $\upsilon_{ij} = 1-(I-1)\alpha$ and $\upsilon_{ik} = \alpha$ for $k \neq j$, and the manager of firm j maximizes $\sum_k \upsilon_{ik}\pi_k$.

[2] Suppose that each investor acquires a share α of those other firms. To compute λ_{jk} for a given $k \neq j$, note that if i is the majority shareholder of j then $\zeta_{ij} = 1-(I-1)\alpha$ and $\upsilon_{ik} = \alpha$; if i' is the majority shareholder of k, then $\zeta_{i'j} = \alpha$ and i' receives an own-firm profit share of $\upsilon_{i'k} = 1-(I-1)\alpha$. Finally, there are $I-2$ investors who are minority shareholders of j and k; for these investors, the product of their profit shares (and control) is equal to α^2. This explains the numerator of λ_{jk}. The denominator follows similarly and we obtain the expression for λ^{PC}.

symmetric is, for $\alpha < 1/(n-1)$, $\theta_{jj} = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha](\alpha+1)}$ and $\theta_{jk} = \frac{\alpha}{[1-(n-1)\alpha](\alpha+1)}$ for all j and all $j \neq k$.[3] Hence, the profit of firm j with symmetric stakes is given by

$$\phi_j = \frac{1-(n-2)\alpha}{[1-(n-1)\alpha]\,(\alpha+1)}\pi_j + \frac{\alpha}{[1-(n-1)\alpha]\,(\alpha+1)}\sum_{k\neq j}\pi_k.$$

Maximizing the above expression is equivalent to maximizing $\pi_j + \lambda\sum_{k\neq j}\pi_k$, where $\lambda = \lambda^{\text{CO}} \equiv \alpha/\left[1-(n-2)\alpha\right]$.

Comparative statics. The results for λ^{SFI} and λ^{CO} follow by inspection. Regarding the case of proportional control, we have that

$$\frac{\partial\lambda^{PC}}{\partial I} = \frac{\alpha^2\left[\alpha^2 I^2 - 4\alpha I + 3\right]}{\left(\alpha^2 I^2 - \alpha^2 I - 2\alpha I + 2\alpha + 1\right)^2}; \quad \frac{\partial\lambda^{PC}}{\partial\alpha} = \frac{2\,(1-\alpha I)}{\left(\alpha^2 I^2 - \alpha^2 I - 2\alpha I + 2\alpha + 1\right)^2}.$$

Therefore, $\partial\lambda^{PC}/\partial I > 0$ iff $\rho_{PC}(\alpha) = \alpha^2\left(\alpha^2 I^2 - 4\alpha I + 3\right) > 0$ for any $I \geq 2$ and $\alpha < 1/I$. Solving for $\rho_{PC}(\alpha) = 0$, the quadratic $(\alpha^2 I^2 - 4\alpha I + 3)$ gives the solutions $\alpha = 1/I$ and $\alpha = 3/I$. For $\alpha \in (0, 1/I)$, $(\alpha^2 I^2 - 4\alpha I + 3) > 0$ and $\alpha^2 > 0$ and, thus, $\rho_{PC}(\alpha) > 0$.

By differentiating with respect to α, we obtain $\rho'_{PC} = 4\alpha\left(\alpha^2 I^2 - 3\alpha I + 3/2\right) > 0$ for $\alpha \in (0, 1/I)$. Therefore, $\rho_{PC} > 0$ for $\alpha \in (0, 1/I)$ since $\rho_{PC}(0) = 0$.

Clearly, $\partial\lambda^{PC}/\partial\alpha > 0$ for $\alpha < 1/I$.

Ranking. Let us compare λ^{SFI} and λ^{PC}; after simplifying we obtain

$$\lambda^{SFI} - \lambda^{PC} = \frac{\alpha(1-\alpha I)}{-\left[1-\alpha(I-1)\right]\left[1 + I(I-1)\alpha^2 - 2(I-1)\alpha\right]}.$$

For $\alpha < 1/I$, we have $\lambda^{SFI} < \lambda^{PC}$ iff $\rho_{SP}(\alpha) = 1 + I(I-1)\alpha^2 - 2(I-1)\alpha > 0$. Note that $\rho_{SP}(0) = 1 > 0$, furthermore $\rho'_{SP}(\alpha) = 2I(I-1)\alpha - 2(I-1) = 2(I-1)(I\alpha - 1) < 0$. Since $\rho''_{SP}(\alpha) = 2(I-1)I > 0$, the global minimum is located at $\alpha = 1/I$, at which $\rho_{SP}(1/I) = 1/I > 0$. Thus, $\rho_{SP}(\alpha) > 0$ and as a result $\lambda^{SFI} < \lambda^{PC}$.

Finally, for $n = I$

$$\lambda^{SFI} - \lambda^{CO} = \frac{\alpha^2}{\left[-1 + (I-2)\alpha\right]\left[-1 + (I-1)\alpha\right]},$$

thus $\lambda^{SFI} - \lambda^{CO} > 0$ for $\alpha < 1/I$, hence $\lambda^{PC} > \lambda^{SFI} > \lambda^{CO}$.

Table 4: Summary of Basic Expressions at the Symmetric Equilibrium of the Simultaneous Game

Second-Order Conditions	
$\partial_{q_iq_i}\phi_i = (\partial^2\phi_i/\partial q_i^2)\big\|_{q^*,x^*} =$	$f'(Q^*)(2+\delta\Lambda/n) < 0$
$\partial_{x_ix_i}\phi_i = (\partial^2\phi_i/\partial x_i^2)\big\|_{q^*,x^*} =$	$-(c''(Bx^*)\tilde{\lambda}q^* + \Gamma''(x^*)) < 0$
$(\partial_{q_iq_i}\phi_i)(\partial_{x_ix_i}\phi_i) - (\partial_{x_iq_i}\phi_i)^2 =$	$-f'(Q^*)(2+\Lambda\delta/n)[c''(Bx^*)(Q^*/n)\tilde{\lambda} + \Gamma''(x^*)] - c'(Bx^*)^2 > 0$
Cross-Derivatives	
$\partial_{q_iq_j}\phi_i = (\partial^2\phi_i/\partial q_i\partial q_j)\big\|_{q^*,x^*} =$	$f'(Q^*)(1+\lambda+\delta\Lambda/n) < (>) 0$ for $\delta > (<) -(1+\lambda)n/\Lambda$
$\partial_{x_ix_j}\phi_i = (\partial^2\phi_i/\partial x_i\partial x_j)\big\|_{q^*,x^*} =$	$-c''(Bx^*)\beta q^*\{1+\lambda[1+(n-2)\beta]\} < 0$ for $\beta c'' > 0$
$\partial_{x_iq_i}\phi_i = (\partial^2\phi_i/\partial x_i\partial q_i)\big\|_{q^*,x^*} =$	$-c'(Bx^*) > 0$
$\partial_{\lambda q_i}\phi_i = (\partial^2\phi_i/\partial\lambda\partial q_i)\big\|_{q^*,x^*} =$	$f'(Q^*)(n-1)q^* < 0$
$\partial_{\lambda x_i}\phi_i = (\partial^2\phi_i/\partial\lambda\partial x_i)\big\|_{q^*,x^*} =$	$-\beta(n-1)c'(Bx^*)q^* > 0$ for $\beta > 0$
Regularity Conditions	
$\Delta_q \equiv \partial_{q_iq_i}\phi_i + \partial_{q_iq_j}\phi_i(n-1) =$	$f'(Q^*)[n+\Lambda(\delta+1)] < 0$
$\Delta_x \equiv \partial_{x_ix_i}\phi_i + \partial_{x_ix_j}\phi_i(n-1) =$	$-(c''(Bx^*)B\tau q^* + \Gamma''(x^*)) < 0$
$\Delta \equiv$	$\Delta_q\Delta_x - [\partial_{x_iq_i}\phi_i + \beta(n-1)\partial_{x_iq_i}\phi_i][\partial_{x_iq_i}\phi_i + \lambda(n-1)\beta\partial_{x_iq_i}\phi_i] = \Delta_q\Delta_x - (\partial_{x_iq_i}\phi_i)^2\tau B > 0$

with $B \equiv 1+\beta(n-1)$, $\Lambda \equiv 1+\lambda(n-1)$, $\tau \equiv 1+\lambda(n-1)\beta$ and $\tilde{\lambda} \equiv 1+\lambda(n-1)\beta^2$.

Remark: $\Delta_q < 0 \Leftrightarrow \delta\Lambda > -(\Lambda+n)$, whereas $\partial_{q_iq_i}\phi_i < 0 \Leftrightarrow \delta\Lambda > -2n$, thus $\Delta_q < 0$ implies that $\partial_{q_iq_i}\phi_i < 0$,

and to have $\Delta_x < 0$ we need that $c'' > 0$ or $\Gamma'' > 0$, and therefore $\partial_{x_ix_i}\phi_i < 0$.

The signs of the expressions follow under our assumptions.

A.1.2 Simultaneous model

Second order and regularity conditions. To start with, note that

$$\Delta(Q^*, x^*) = -\left[c''(Bx^*)B\tau(Q^*/n) + \Gamma''(x^*)\right]\left[f'(Q^*)(\Lambda(1+\delta)+n)\right] - (c'(Bx^*))^2\tau B > 0. \quad (20)$$

In particular, the above condition can be rewritten as $[\Lambda(1+\delta)+n]\,H(\beta) - \tau B > 0$. Second order conditions are: (i) $\partial_{q_iq_i}\phi_i < 0$, since $\partial_{q_iq_i}\phi_i = 2f'(Q) + \Lambda(Q/n)f''(Q) = f'(Q)(2+\Lambda\delta/n)$, we have $\partial_{q_iq_i}\phi_i < 0$ if $\delta > -2n/\Lambda$, which is implied by assumption $\Delta_q < 0$; (ii) $\partial_{x_ix_i}\phi_i < 0$, which is trivially satisfied by Assumptions A.2 and A.3; and (iii) $\partial_{q_iq_i}\phi_i\,(\partial_{x_ix_i}\phi_i) - (\partial_{q_ix_i}\phi_i)^2 > 0$, which is equivalent to

$$c'(Bx^*)^2 + f'(Q^*)(2+\Lambda\delta/n)\left[c''(Bx^*)(Q^*/n)\tilde{\lambda} + \Gamma''(x^*)\right] < 0, \quad (21)$$

[3] See Vives (1999, pp. 145-147) for a solution of a formally identical problem. Gilo et al. (2006, Lemma 1, p.85) also show that $\theta_{jj} \geq 1$ for all j, and $0 \leq \theta_{jk} < \theta_{jj}$ for all j and all $j \neq k$.

where $\tilde{\lambda} = 1 + \lambda(n-1)\beta^2$. Noting that $\partial_{q_i q_j}\phi_i = f'(Q^*)(1+\lambda) + f''(Q^*)\Lambda q^* = f'(Q^*)(1+\lambda+\delta\Lambda/n)$, we have that

$$\Delta_q \equiv \partial_{q_i q_i}\phi_i + \partial_{q_i q_j}\phi_i(n-1) = f'(Q^*)\left[n + \Lambda(\delta+1)\right] < 0,$$

which is satisfied if $\delta > -(n+\Lambda)/\Lambda$. Similarly, noting that $\partial_{x_i x_i}\phi_i = -c''(Bx^*)\tilde{\lambda}q^* - \Gamma''(x^*)$ and $\partial_{x_i x_j}\phi_i = -c''(Bx^*)\beta q^*\left\{1 + \lambda\left[1 + (n-2)\beta\right]\right\}$, it is straightforward to show that

$$\Delta_x \equiv \partial_{x_i x_i}\phi_i + \partial_{x_i x_j}\phi_i(n-1) = -\left[c''(Bx^*)B\tau q^* + \Gamma''(x^*)\right] < 0,$$

which is satisfied by Assumptions A.2 and A.3.

Proof of Lemma 1. Using equation (6) and Table 4 we obtain

$$\frac{\partial x^*}{\partial \lambda} = \frac{c'(Bx^*)f'(Q^*)(n-1)q^*}{\Delta}\left\{\beta\left[\Lambda(1+\delta)+n\right] - \tau\right\}.$$

Since $\Delta > 0$,

$$\begin{aligned}\operatorname{sign}\left\{\frac{\partial x^*}{\partial \lambda}\right\} &= \operatorname{sign}\left\{\beta\left[\Lambda\left(1+\delta\right)+n\right]-\tau\right\} \\ &= \operatorname{sign}\left\{\beta - \frac{\tau}{\Lambda\left(1+\delta\right)+n}\right\} = \operatorname{sign}\left\{\beta - P'(c)\frac{\tau}{n}\right\},\end{aligned}$$

where $P'(c) = n/[\Lambda(1+\delta)+n]$. Note that $\Lambda(1+\delta)+n > 0$ since $\Delta_q < 0$. Finally, by substituting

$$\operatorname{sign}\left\{\beta\left[\Lambda\left(1+\delta\right)+n\right]-\tau\right\} = \operatorname{sign}\left\{\beta(1+n+\delta\Lambda)-1\right\}.\blacksquare$$

Proof of Corollary 1. From Lemma 1 we have that if $\delta \leq -(1+n)/\Lambda$, so $1+n+\delta\Lambda \leq 0$, then $\partial x^*/\partial\lambda < 0$, which, using equation (7), in turn implies that $\partial q^*/\partial\lambda < 0$: for all β only R_{I} exists. If $\delta > -(n+1)/\Lambda$, then in addition to R_{I}, region R_{II} exists only if $\delta > -n/\Lambda$ also holds. The reason is that when $1+n+\delta\Lambda > 0$, then, from Lemma 1, $\partial x^*/\partial\lambda > 0$ requires that $\beta > 1/(1+n+\delta\Lambda)$. However, $1/(1+n+\delta\Lambda) < 1$ only if $\delta > -n/\Lambda$, in which case there exists some region of feasible spillover values for which $\partial x^*/\partial\lambda > 0$. Note that for a given n, the condition $\delta > -n/\Lambda$ is stricter than the condition $\delta > -(n+1)/\Lambda$. Thus, for $\delta \leq -n/\Lambda$ only R_{I} exists, and since $-n/\Lambda$ increases with λ, the result holds for any λ if $\delta \leq -n$.$\blacksquare$

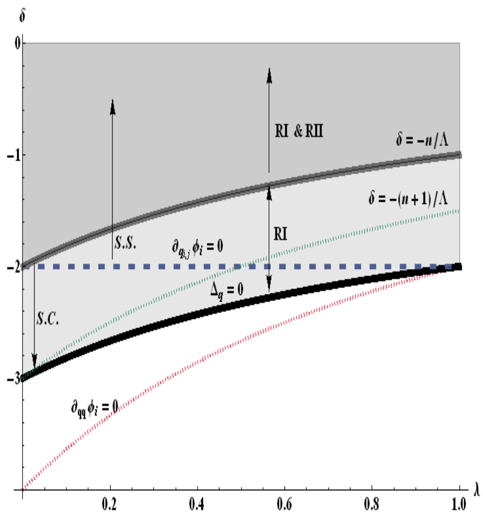

Fig. 5a. $n = 2$.

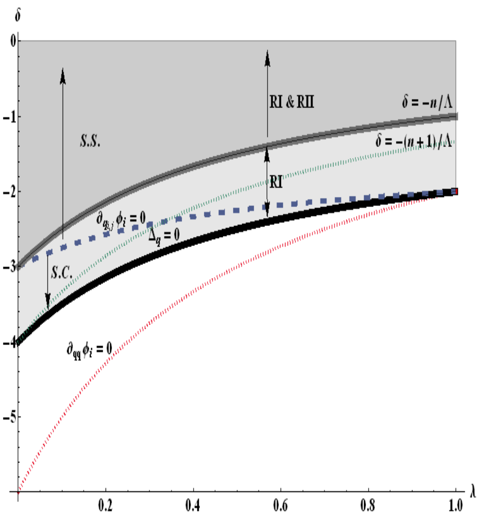

Fig. 5b. $n = 3$.

Fig. 5. Existence of regions R_{I} and R_{II} with second-order, stability and strategic complements/substitutes output competition conditions.

Proof of Lemma 2. If we totally differentiate the first order conditions (FOCs) and solve for $\partial q^*/\partial\lambda$, we obtain

$$\frac{\partial q^*}{\partial \lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\beta c'(Bx^*)^2 \left\{ B + \frac{f'(Q^*)}{\beta c'(Bx^*)^2} \left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*) \right] \right\}.$$

Let $H \equiv \beta \left(\partial_{\lambda q_i}\phi_i/\partial_{\lambda x_i}\phi_i\right)\left(\Delta_x/\partial_{x_i q_i}\phi_i\right) = -\left(f'(Q^*)/c'(Bx^*)^2\right)\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]$, evaluated at the equilibrium (Q^*, x^*). From the requirement that $c'' > 0$ or $\Gamma'' > 0$ we obtain that $\lim_{\beta\to 0} H/\beta = \infty$. H is continuous in β as long as $Q^*(\beta), x^*(\beta)$ are since all the functions involved in the definition of H are continuous and $c' < 0$. We have that $Q^*(\beta), x^*(\beta)$ are in fact differentiable given our assumptions (see the proof of Proposition 3). The above expression can be rewritten as

$$\frac{\partial q^*}{\partial \lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\beta \left(c'(Bx^*)\right)^2 \left(B - \frac{H}{\beta}\right), \tag{22}$$

thus sign $\{\partial q^*/\partial\lambda\} =$ sign $\{\beta B - H\}$.■

Proof of Corollary 2. Under A.4 and Lemma 2, $\partial q^*/\partial\lambda > 0$ (so R_{III} exists) if $\beta > \beta'$. We now show that the condition $n > H(1)$ guarantees that $\beta' < 1$. First, note that $\lim_{\beta\to 0} H/\beta = \infty$ (when $c'' > 0$ or $\Gamma'' > 0$), while $B = 1$ at $\beta = 0$. Since $H(\beta)/\beta$ is downward sloping, by continuity there exists only one value for $\beta (= \beta')$ at which $H(\beta) = \beta B$. If the condition $H(\beta) < \beta B$ holds at $\beta = 1$ (which is equivalent to the condition $n > H(1)$), then necessarily H/β intersects B at some β less than 1, thus $\beta' < 1$.■

Proof of Proposition 2. Profit per firm as a function of λ at equilibrium is given by

$$\pi^*(\lambda) = (f(Q^*) - c(Bx^*))\,q^* - \Gamma(x^*).$$

By differentiating π^* with respect to λ, we obtain

$$\pi^{*\prime}(\lambda) = f'(Q^*)n\frac{\partial q^*}{\partial \lambda}q^* - c'(Bx^*)B\frac{\partial x^*}{\partial \lambda}q^* + (f(Q^*) - c(Bx^*))\frac{\partial q^*}{\partial \lambda} - \Gamma'(x^*)\frac{\partial x^*}{\partial \lambda}.$$

Using that in equilibrium $f(Q^*) - c(Bx^*) = -f'(Q^*)\Lambda q^*$ and $\Gamma'(x^*) = -c'(Bx^*)q^*\tau$, we can rewrite the above expression as

$$\begin{aligned}
\pi^{*\prime}(\lambda) &= f'(Q^*)n\frac{\partial q^*}{\partial \lambda}q^* - c'(Bx^*)B\frac{\partial x^*}{\partial \lambda}q^* - f'(Q^*)\Lambda q^*\frac{\partial q^*}{\partial \lambda} + c'(Bx^*)q^*\tau\frac{\partial x^*}{\partial \lambda} \\
&= f'(Q^*)(n-\Lambda)q^*\frac{\partial q^*}{\partial \lambda} + c'(Bx^*)(\tau - B)q^*\frac{\partial x^*}{\partial \lambda} \\
&= (n-1)(1-\lambda)q^*\left(f'(Q^*)\frac{\partial q^*}{\partial \lambda} - \beta c'(Bx^*)\frac{\partial x^*}{\partial \lambda}\right).
\end{aligned}$$

In R_{II}, we have that $\partial x^*/\partial\lambda > 0$ and $\partial q^*/\partial\lambda < 0$. Hence from the above expression it is clear that $\pi^{*\prime}(\lambda) > 0$. Note also that when $\beta = 0$, the equilibrium is in R_{I}, and therefore $\pi^{*\prime}(\lambda) > 0$ since $\partial q^*/\partial\lambda < 0$. To determine sign$\{\pi^{*\prime}(\lambda)\}$ in R_{I} and R_{III} for $\beta > 0$, we replace $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$ with the expressions derived in the proofs of Lemmata 1, 2:

$$\begin{aligned}
\pi^{*\prime}(\lambda) &= (n-1)(1-\lambda)q^*\left(f'(Q^*)\frac{(n-1)q^*}{\Delta}c'(Bx^*)^2\beta\left(B - \frac{H(\beta)}{\beta}\right)\right. \\
&\qquad \left. -\beta c'(Bx^*)\frac{(n-1)q^*}{\Delta}f'(Q^*)c'(Bx^*)\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}\right) \\
&= \vartheta_\pi\left\{\beta\left[\Lambda(1+\delta)+n\right] - \tau + \frac{H(\beta)}{\beta} - B\right\},
\end{aligned}$$

where $\vartheta_\pi \equiv (n-1)(1-\lambda)q^*\left[(n-1)q^*/\Delta\right]c'(Bx^*)^2\beta(-f'(Q^*))$ is positive. Therefore,

$$\operatorname{sign}\left\{\pi^{*\prime}(\lambda)\right\} = \operatorname{sign}\left\{(n+1+\delta\Lambda)\beta - 1 + \frac{H(\beta)}{\beta} - B\right\}, \tag{23}$$

so it follows that $\pi^{*\prime}(\lambda) > 0$ if

$$1 - (n+1+\delta\Lambda)\beta < \frac{H(\beta)}{\beta} - B, \text{ or equivalently} \tag{24}$$

$$2(1-\beta) - \delta\Lambda\beta < \frac{H(\beta)}{\beta}. \tag{25}$$

From Table 4 and using that in equilibrium $\tau q^* = -\Gamma'(x^*)/c'(Bx^*)$, the regularity condition can be written as

$$-\left(-c''(Bx^*)B\frac{\Gamma'(x^*)}{c'(Bx^*)}+\Gamma''(x^*)\right)\frac{f'(Q^*)}{c'(Bx^*)^2}\left[\Lambda\left(1+\delta\right)+n\right]-\tau B>0.$$

Noting that

$$H(\beta)=\frac{-f'(Q^*)}{c'(Bx^*)^2}\left(-\frac{c''(Bx^*)}{c'(Bx^*)}B\Gamma'(x^*)+\Gamma''(x^*)\right),$$

we can rewrite the regularity condition in terms of H as follows: $\left[\Lambda(1+\delta)+n\right]H(\beta)-\tau B>0$, with $\Lambda(1+\delta)+n>0$ since $\Delta_q<0$. Thus, if the equilibrium is regular:

$$\frac{H(\beta)}{\beta}>\frac{\tau B}{\left[\Lambda(1+\delta)+n\right]\beta}.$$

Then, we only have to show that:

$$\tilde{g}(\beta)\equiv\tau B>\tilde{h}(\beta)\equiv[2(1-\beta)-\Lambda\delta\beta][\Lambda(1+\delta)+n]\beta$$

holds. Note that $\tilde{g}(0)=1$, $\tilde{g}'(\beta)>0$, $\tilde{g}''(\beta)>0$ for $\beta>0$ and $\tilde{g}''(0)=0$. On the other hand, $\tilde{h}(0)=0$ and

$$\tilde{h}'(\beta)=2[\Lambda(1+\delta)+n][1-(2+\Lambda\delta)\beta].$$

Furthermore, it can be shown that solving the equation $\tilde{g}(\beta)=\tilde{h}(\beta)$ for β yields the following two roots:

$$\beta_1=\frac{1}{\Lambda\delta+n+1}\text{ and }\beta_2=\frac{1}{\Lambda(\delta+1)+1}.$$

Consider R_{I}. If the smallest (positive) root in this region is larger or equal to the spillover threshold $\underline{\beta}$ that determines R_{I} (i.e. for $\beta<\underline{\beta}$, $\frac{\partial x^*}{\partial\lambda}<0$ for all λ), then $\tilde{g}(\beta)>\tilde{h}(\beta)$ in R_{I}, and consequently, $\pi^{*\prime}(\lambda)>0$. First, note that when $\Lambda\delta+n+1>0$, $\beta_1=\underline{\beta}$. We distinguish the following cases:

- If $\Lambda\delta+n+1>0$, then: if $\Lambda(\delta+1)+1>0$, (for $\lambda<1$) $\beta_2>\beta_1=\underline{\beta}>0$, while if $\Lambda(\delta+1)+1<0$, $\beta_1=\underline{\beta}>0>\beta_2$, so in any case $\beta_1=\underline{\beta}$ is the smallest positive root in the region and, thus, $\tilde{g}(\beta)>\tilde{h}(\beta)$ for $\beta\in(0,\beta_1)$. Also, in any case for $\beta=\beta_1$, $\partial x^*/\partial\lambda=0$ and so $\mathrm{sign}\{\pi^{*\prime}(\lambda)\}=\mathrm{sign}\{f'(Q^*)\partial q^*/\partial\lambda\}$, which is positive in R_{I} since in this region: $\partial q^*/\partial\lambda<0$.

- If $\Lambda\delta+n+1<0$, then $0>\beta_1>\beta_2$ (for $\lambda<1$) and $\tilde{h}''(\beta)>0$, so $\tilde{g}(\beta)>\tilde{h}(\beta)$ for all β.

Now consider R_{III}, which may exist only if $\delta>-n/\Lambda$, in which case $\beta_1>0$. Furthermore, $\beta'\geq\beta_1$. We show that for any $\beta>\beta'$, $\tilde{g}(\beta)>\tilde{h}(\beta)$. We distinguish the following cases:

- If $\delta > -2/\Lambda$, then $\beta_2 > \beta_1 > 0$ (for $\lambda < 1$) and $\tilde{h}''(\beta) < 0$. Hence, $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $\beta > \beta_2$. Thus, it suffices to show that $\beta' > \beta_2$. Note that if $\pi'(\lambda) > 0$ for $\beta = \beta'$, then necessarily $\beta' > \beta_2$ since $\beta' > \beta_1$ and $\pi'(\lambda) < 0$ for $\beta \in (\beta_1, \beta_2)$. Since condition (24) holds at $\beta = \beta'$: $H(\beta')/\beta' - \left[1 + \beta'(n-1)\right] = 0 > 1 - (n + 1 + \delta\Lambda)\beta$, we have $\beta' > \beta_2$.

- If $-(\Lambda+1)/\Lambda < -n/\lambda < \delta < -2/\Lambda$ or $-n/\lambda < -(\Lambda+1)/\Lambda < \delta < -2/\Lambda$, then $\beta_2 > \beta_1 > 0$ (for $\lambda < 1$) and $\tilde{h}''(\beta) > 0$, so we can conclude that $\beta' > \beta_2$.

- If $-n/\lambda < \delta < -(\Lambda + 1)/\Lambda < -2/\Lambda$, then $\beta_1 > 0$, $\beta_2 < 0$ and $\tilde{h}''(\beta) > 0$, so $\tilde{g}(\beta) > \tilde{h}(\beta)$ only for $\beta < \beta_1$. Also, for $\beta' < 1$, condition (24) holds, so $\tilde{g}(\beta) > \tilde{h}(\beta)$ for $\beta \geq \beta'$. But then it should be $\beta' < \beta_1$, a contradiction, so in this case R_{III} does not exist.■

Proof of Proposition 3. By totally differentiating the two FOCs with respect to β, we obtain

$$\frac{\partial q^*}{\partial \beta} = \frac{1}{\Delta}\left[\left(\partial_{\beta x_i}\phi_i\right)\left(\partial_{x_i q_i}\phi_i\right) B - \left(\partial_{\beta q_i}\phi_i\right)\Delta_x\right] \tag{26}$$

$$\frac{\partial x^*}{\partial \beta} = \frac{1}{\Delta}\left[\left(\partial_{\beta q_i}\phi_i\right)\left(\partial_{x_i q_i}\phi_i\right) \tau - \left(\partial_{\beta x_i}\phi_i\right)\Delta_q\right]. \tag{27}$$

Since $\partial_{x_i q_i}\phi_i > 0$ and $\partial_{\beta q_i}\phi_i > 0$, $\Delta > 0$, $\Delta_x < 0$ and $\Delta_q < 0$, the sign of the impact of β on output and R&D in equilibrium depends on the sign of $\partial_{\beta x_i}\phi_i$. It can be shown that

$$\partial_{\beta x_i}\phi_i = -c'(Bx^*)\frac{(n-1)q^*}{B}\tau\left(\frac{\lambda B}{\tau} - \chi(Bx^*)\right) \tag{28}$$

and the result follows.■

Proof of Proposition 4. To prove Proposition 4 a few preliminary lemmata (assuming A.1-A.4) are useful.

LEMMA 5 *Suppose that $\delta > -2$, then for given λ, $W'(\lambda) > 0$ iff $\beta > \hat{\beta}\ (\lambda)$ where $\hat{\beta}$ is the unique positive solution to the equation*

$$\frac{H(\beta)}{\beta} - B = \left[(n - \Lambda)/\Lambda\right]\left[(1 + n + \delta\Lambda)\beta - 1\right]. \tag{29}$$

Proof. We first derive the condition that determines $\hat{\beta}$. By inserting $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$

(given in proofs of Lemmata 1 and 2) into (9) we obtain:

$$\begin{aligned}
W'(\lambda) &= -\Lambda f'(Q^*)\frac{(n-1)q^*}{\Delta}c'(Bx^*)^2\beta\left(B-\frac{H(\beta)}{\beta}\right)Q^* \\
&\quad -(1-\lambda)\beta(n-1)c'(Bx^*)\frac{(n-1)q^*}{\Delta}f'(Q^*)c'(Bx^*)\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}Q^*, \\
&= \vartheta_w\left(\Lambda\left(B-\frac{H(\beta)}{\beta}\right)+(1-\lambda)(n-1)\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}\right),
\end{aligned}$$

where $\vartheta_w \equiv [(n-1)q^*/\Delta]c'(Bx^*)^2(-f'(Q^*))\beta Q^*$ is positive. Note that $(1-\lambda)(n-1) = n-\Lambda$, thus for $\beta > 0$, $W'(\lambda) > 0$ iff

$$\frac{H(\beta)}{\beta} - B < \frac{n-\Lambda}{\Lambda}\left[(1+n+\delta\Lambda)\beta - 1\right]. \tag{30}$$

Note that $\lim_{\beta\to 0} H/\beta = \infty$ and (by Assumption A.4) the left-hand side of (30) is decreasing in β. The right-hand side of (30) is increasing in β (since $1+n+\delta\Lambda > 0$ holds when R_{II} and R_{III} exist) and finite at $\beta = 0$. Thus, there exists a unique positive threshold $\hat{\beta}$ that solves the equation (29), and for any $\beta > \hat{\beta}$ condition (30) holds, that is, $W'(\lambda) > 0$.■

LEMMA 6 *We have that $\hat{\beta}(\lambda) < \beta'(\lambda)$ for all λ, which implies that $\bar{\beta} \equiv \hat{\beta}(0) < \beta'(0)$. Furthermore, $\bar{\beta} < 1$ if*

$$n + (n-1)(\delta + n) - H(1) > 0. \tag{31}$$

Proof. We first show that $\beta'(\lambda) > \hat{\beta}(\lambda)$ for any λ, and as a result $\beta'(0) > \bar{\beta} \equiv \hat{\beta}(0)$. Suppose that for given λ, $\hat{\beta} > \beta'$, then from Lemma 2 we have that for $\beta \in (\beta', \hat{\beta})$ it holds that $\partial q^*/\partial\lambda > 0$. Thus, from equation (7) it also holds that $\partial x^*/\partial\lambda > 0$, which given equation (9) implies that $W'(\lambda) > 0$. However, we have that $W'(\lambda) < 0$ for $\beta < \hat{\beta}$, a contradiction. Suppose now that $\hat{\beta} = \beta'$, then we can pick β such that $\beta = \hat{\beta} = \beta'$, and since $H - \beta B|_{\beta=\beta'} = 0$, from equation (29) we have that $\hat{\beta} = \beta' = 1/(1+n+\delta\Lambda)$, which implies that $\partial x^*/\partial\lambda = 0$ (see proof of Lemma 1), and from equation (7) this in turn implies that $\partial q^*/\partial\lambda < 0$. However, at $\beta = \beta'$, $\beta B - H = 0$, so $\partial q^*/\partial\lambda = 0$, a contradiction.

The proof of Lemma 5 shows that $W'(\lambda) > 0$ for some λ if $\beta > \hat{\beta}(\lambda)$, where $\hat{\beta}$ is the unique positive solution to the equation (29). Furthermore, $\hat{\beta} < 1$ if condition (30) evaluated at $\beta = 1$ holds. Therefore, by evaluating (30) at $\lambda = 0$ and $\beta = 1$ we obtain that condition (31) ensures that $\bar{\beta} < 1$.■

We turn now to prove successively each of the statements of Proposition 4. Let $\delta > -2$:

i) $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 0$ if $\beta \leq \bar{\beta}$. First, we show that there does not exist $\beta < \bar{\beta}$ such that $W'(\lambda) > 0$ for some positive λ. This follows trivially from the assumption that $W(\lambda)$ is single peaked: since for any $\beta \leq \bar{\beta}$, $W'(0) \leq 0$, we have that $W'(\lambda) < 0$ for all positive λ, otherwise there would exist another stationary point that is a (local) minimum, a contradiction. In addition, if $\beta \leq \bar{\beta}$, then $\lambda^o_{\text{CS}} = 0$: from Lemma 6 we know that $\beta'(\lambda) > \bar{\beta} = \hat{\beta}(0)$ for all λ. For $\beta \leq \bar{\beta}$ we have then that $CS'(\lambda) < 0$ for all λ, thus $\lambda^o_{\text{CS}} = 0$.

ii) $\lambda^o_{\text{TS}} > \lambda^o_{\text{CS}} = 0$ if $\beta \in (\bar{\beta}, \beta'(0))$. Since $\bar{\beta} = \hat{\beta}(0)$, the result that $\lambda^o_{\text{TS}} > 0$ for $\beta > \bar{\beta}$ follows immediately from Lemma 5 because then $W'(0) > 0$. In addition, $\beta < \beta'(0)$ yields $\lambda^o_{\text{CS}} = 0$: when H is weakly increasing in λ, $\beta'(\lambda)$ also is, and consequently if $\beta < \beta'(0)$, then $\beta < \beta'(\lambda)$ for all λ, i.e., $\partial q^*/\partial\lambda < 0$ for all λ, thus $\lambda^o_{\text{CS}} = 0$.

iii) We first show that $\lambda^o_{\text{TS}} > 0$ and $\lambda^o_{\text{CS}} > 0$ if $\beta > \beta'(0)$. From Lemma 6 it follows that $\beta > \beta'(0) > \bar{\beta}$, which yields $\lambda^o_{\text{TS}} > 0$. If $\beta > \beta'(0)$, Lemma 2 implies that $\partial q^*/\partial\lambda > 0$ at $\lambda = 0$, which implies that $CS'(0) > 0$, and therefore $\lambda^o_{\text{CS}} > 0$.

Next we show that $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$ when H is weakly increasing in λ. Note that $B > H/\beta$ at $\lambda = 0$. Since H is weakly increasing in λ, for a given β, we may face the following three cases: 1) for all λ, $B > H/\beta$; 2) there exists an interval (which could be a singleton) $L \subset (0, 1]$ at which $H/\beta = B$ but $H/\beta \leq B$ for $\lambda < 1$; 3) there exists an interval of values of λ, $L \subset (0, 1]$ at which $H/\beta = B$ but $H/\beta > B$ for some $\lambda < 1$. In all three cases $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$:

Case 1: Here, $\partial q^*/\partial\lambda > 0$ and, by (7), $\partial x^*/\partial\lambda > 0$ for all λ, which from equation (9) yields $W'(\lambda) > 0$ for all λ; thus $\lambda^o_{\text{TS}} = \lambda^o_{\text{CS}} = 1$.

Cases 2 and 3: In these two cases, in the region of values for λ where $H/\beta = B$ we have $\partial q^*/\partial\lambda = 0$ ($CS'(\lambda) = 0$), while $\partial x^*/\partial\lambda > 0$, consequently $W'(\lambda) > 0$. It follows that in Case 2, $\lambda^o_{\text{TS}} = 1$, while any $\lambda \in L$ is optimal in terms of CS since $\partial q^*/\partial\lambda > 0$ for any $\lambda < \min L$, thus $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$; in Case 3, any $\lambda \in L$ is optimal in terms of CS since $\partial q^*/\partial\lambda < 0$ for $\lambda > \max L$; $\lambda^o_{\text{TS}} \geq \max L$ since $W'(\lambda) > 0$ for lower values of λ; as a result $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}}$.

The particular case where $\beta = \beta'(0)$ can be dealt with similarly to obtain that $\lambda^o_{\text{TS}} \geq \lambda^o_{\text{CS}} \geq 0$.

Finally, we show that λ^o_{TS} and λ^o_{CS} are strictly increasing in β when λ^o_{TS} and λ^o_{CS} are in $(0, 1)$. We have that

$$W'(\lambda) = -c'(Bx^*)^2 f'(Q^*)) \frac{(n-1)q^*}{\Delta} \beta Q^* \varphi(\lambda, \beta)$$

where $\varphi(\lambda, \beta) = \Lambda(B - H/\beta) + (n - \Lambda)(\beta(1 + n + \delta\Lambda) - 1)$. Consider the FOC of the welfare maximizing problem, $W'(\lambda) = 0$ if and only if $\varphi(\lambda, \beta) = 0$. Given single-peakedness of W, $\text{sign}\{d\lambda^o_{\text{TS}}/d\beta\} = \text{sign}\{\partial\varphi(\lambda^o_{\text{TS}}, \beta)/\partial\beta\}$. We have that

$$\frac{\partial\varphi}{\partial\beta} = \left[\Lambda \frac{\partial}{\partial\beta}\left(B - \frac{H(\beta)}{\beta}\right) + (n - \Lambda)(1 + n + \delta\Lambda)\right] > 0$$

since $H(\beta)/\beta$ is downward sloping, $n-\Lambda \geq 0$, and interior optimal lambdas require that R_{II} exists, i.e., $\delta > -n/\Lambda$, which in turn implies that $1+n+\delta\Lambda > 0$. Similarly, we can show, using the fact that $H(\beta)/\beta$ is decreasing in β, the result for $\lambda^o_{\text{CS}} \in (0,1)$.■

Proof of Proposition 5. If $\delta > -(1+n)/n$, then $1+n+\delta\Lambda > 0$ for all λ. From Lemma 1 we know that when $\beta \leq 1/(1+n+\delta\Lambda)$: $\partial x^*/\partial\lambda \leq 0$. From Lemma 5 we have that $W'(\lambda) > 0$ if $\beta > \hat{\beta}\ (\lambda)$ where $\hat{\beta}$ is given in Lemma 5. Necessarily, $\hat{\beta} > 1/(1+n+\delta\Lambda)$, otherwise for any $\beta \in \left[\hat{\beta}, 1/(1+n+\delta\Lambda)\right]$, we have that $\partial x^*/\partial\lambda \leq 0$, which from equation (7) implies that $\partial q^*/\partial\lambda < 0$, which using equation (9) yields $W'(\lambda) < 0$, a contradiction. Since $\hat{\beta}(\lambda) > 1/(1+n+\delta\Lambda)$ for any λ, then $\hat{\beta}(0) = \bar{\beta} > \underline{\beta}$, and given Lemma 6, $\underline{\beta} < \bar{\beta} < \beta'(0)$ is established. Next we prove each of the statements. (i) When $\delta > -(1+n)/n$ not only R_{I} but also R_{II} may exist for $n \geq 2$ since $\delta > -2$. If $-(1+n)/n < \delta < 0$, then $\inf\{1/(1+n+\Lambda\delta) : \lambda \in [0,1]\} = 1/(1+n+\delta) > 0$, whereas if $\delta \geq 0$, $\inf\{1/(1+n+\Lambda\delta) : \lambda \in [0,1]\} = 1/\left[1+n(1+\delta)\right] > 0$. In both cases, if $\beta \leq \underline{\beta}$, it follows from Proposition 1 that only R_{I} can exist. (ii) Lemma 5 ensures that if $\beta > \bar{\beta} = \hat{\beta}(0)$, then $W'(0) > 0$, thus $\lambda^o_{\text{TS}} > 0$; (iii) From Lemma 2 we have that if $\beta > \beta'(0)$, then $\partial q^*/\partial\lambda|_{\lambda=0} > 0$, which implies that $CS'(0) > 0$: $\lambda^o_{\text{CS}} > 0$.■

A.1.3 Two-stage model

Threshold $\tilde{\beta}(\lambda)$. Let z_i be the action of firm i (q_i in Cournot) and let $\mathbf{z}^*$ be the n-vector of second stage equilibrium actions, then the FOC in the second stage is

$$\frac{\partial}{\partial z_i}\phi_i(\cdot) = 0, \tag{32}$$

whereas in the first stage is

$$\frac{\partial}{\partial x_i}\phi_i(\mathbf{z}^*(\mathbf{x}),\mathbf{x},\lambda) + \sum_{j\neq i}\frac{\partial}{\partial z_j}\phi_i(\mathbf{z}^*(\mathbf{x}),\mathbf{x},\lambda)\frac{\partial}{\partial x_i}z_j^*(\mathbf{x}) = 0, \tag{33}$$

where $\mathbf{x}$ is the n-vector of investment levels. The equilibrium in the two-stage model is thus characterized by the system of equations (32) and (33).

To obtain $\tilde{\beta}(\lambda)$, we first need to obtain the expressions for $\partial z_i^*(\mathbf{x})/\partial x_i$ and $\partial z_j^*(\mathbf{x})/\partial x_i$: we differentiate the FOC (32) with respect to x_i and x_h ($h \neq i$), and evaluate both derivatives in the symmetric equilibrium, then

$$\partial_{z_i z_i}\phi_i(\mathbf{x})\frac{\partial}{\partial x_i}z_i^*(\mathbf{x}) + (n-1)\partial_{z_i z_j}\phi_i(\mathbf{x})\frac{\partial}{\partial x_i}z_j^*(\mathbf{x}) + \partial_{x_i z_i}\phi_i(\mathbf{x}) = 0 \tag{34}$$

and

$$\partial_{z_i z_j}\phi_i(\mathbf{x})\frac{\partial}{\partial x_i}z_i^*(\mathbf{x}) + \left[\partial_{z_i z_i}\phi_i(\mathbf{x}) + (n-2)\partial_{z_i z_j}\phi_i(\mathbf{x})\right]\frac{\partial}{\partial x_i}z_j^*(\mathbf{x}) + \partial_{x_h z_i}\phi_i(\mathbf{x}) = 0. \tag{35}$$

Solving (34) and (35) for $\partial z_i^*(\mathbf{x})/\partial x_i$ and $\partial z_j^*(\mathbf{x})/\partial x_i$ and rearranging terms, we obtain:

$$\frac{\partial}{\partial x_i}z_i^*(\mathbf{x}) = \frac{1}{\Omega}\left[(-\partial_{x_i z_i}\phi_i)\left(\partial_{z_i z_i}\phi_i - \partial_{z_i z_j}\phi_i\right) + (n-1)\,\partial_{z_i z_j}\phi_i\left(\partial_{x_h z_i}\phi_i - \partial_{x_i z_i}\phi_i\right)\right]$$

and

$$\frac{\partial}{\partial x_i}z_j^*(\mathbf{x}) = \frac{1}{\Omega}\left(\partial_{x_i z_i}\phi_i\partial_{z_i z_j}\phi_i - \partial_{x_h z_i}\phi_i\partial_{z_i z_i}\phi_i\right), \tag{36}$$

where

$$\Omega \equiv \left(\partial_{z_i z_i}\phi_i - \partial_{z_i z_j}\phi_i\right)\left[\partial_{z_i z_i}\phi_i + (n-1)\,\partial_{z_i z_j}\phi_i\right]. \tag{37}$$

Consider Cournot competition, $z_i = q_i$. Then, we can rewrite (36) as follows:

$$\frac{\partial}{\partial x_i}q_j^*(\mathbf{x}) = \frac{-c'(Bx)}{\Omega}\partial_{q_i q_i}\phi_i\left(\tilde{\beta}(\lambda) - \beta\right), \tag{38}$$

where

$$\begin{aligned}\Omega &= f'(Q^*)(1-\lambda)\left\{f'(Q^*)\left[n + \Lambda(\delta+1)\right]\right\} \\ &= f'(Q^*)^2(1-\lambda)[n + \Lambda(\delta+1)].\end{aligned}$$

Since $\partial_{q_i q_i}\phi_i = f'(Q^*)(2 + \delta\Lambda/n)$, we have that

$$\begin{aligned}\frac{\partial}{\partial x_i}q_j^*(\mathbf{x}) &= -\frac{c'(Bx)}{f'(Q^*)(1-\lambda)[n + \Lambda(\delta+1)]}\left(2 + \frac{\delta\Lambda}{n}\right)\left(\tilde{\beta}(\lambda) - \beta\right) \\ &= -\frac{c'(Bx)}{nf'(Q^*)(1-\lambda)}\left[\frac{2n + \delta\Lambda}{n + \Lambda(\delta+1)}\right]\left(\tilde{\beta}(\lambda) - \beta\right),\end{aligned}$$

where

$$\tilde{\beta}(\lambda) = \frac{\partial_{q_i q_j}\phi_i}{\partial_{q_i q_i}\phi_i} = \frac{n(1+\lambda) + \Lambda\delta}{2n + \Lambda\delta}$$

with $0 < \tilde{\beta}(\lambda) \leq 1$ and $\tilde{\beta}(1) = 1$. Finally, note that

$$\frac{\partial\phi_i}{\partial q_j} = \Lambda f'(Q^*)q^*(1-\lambda),$$

thus for $\lambda < 1$

$$\begin{aligned}\frac{\partial \phi_i}{\partial q_j}\frac{\partial}{\partial x_i}q_j^*(\mathbf{x}) &= -\Lambda f'(Q^*)q^*(1-\lambda)\frac{c'(Bx)}{nf'(Q^*)(1-\lambda)}\left[\frac{2n+\delta\Lambda}{n+\Lambda(\delta+1)}\right]\left(\tilde{\beta}(\lambda)-\beta\right) \\ &= -c'(Bx)q^*\frac{\Lambda}{n}\left[\frac{2n+\delta\Lambda}{n+\Lambda(\delta+1)}\right]\left(\tilde{\beta}(\lambda)-\beta\right).\end{aligned}$$

When $\lambda = 1$ there is no strategic effect since firms are colluding. In this case maximum joint profits are achieved. However, when $\beta < 1$ the strategic effect does not vanish when $\lambda \to 1$. This is so since $\partial\phi_i/\partial q_j \to 0$ and $\partial q_j^*/\partial x_i \to -\infty$, $i \neq j$, at the same rate when $\lambda \to 1$ and the product of both derivatives is positive in the limit. Suppose that $x_i = x_j$, then as $\lambda \to 1$ it is not efficient to have firm j produce when $\beta < 1$ if x_i increases since then $c_i < c_j$. (Note that cost is linear in output.) When $\beta = 1$ the strategic effect does vanish in the limit $\lambda \to 1$ since then $\tilde{\beta}(\lambda) \to 1$. Indeed, when $\beta = 1$, if x_i increases we have that $c_i = c_j$ and both firms are equally efficient. The consequence is that when $\beta < 1$ and $\lambda \to 1$ there is a discontinuity in total profits at our symmetric equilibrium, not attaining the cartel profits achieved when $\lambda = 1$.

Proof of Lemma 3. We have that

$$\tilde{\beta}(\lambda) = \frac{n(1+\lambda)+\Lambda\delta}{2n+\Lambda\delta}.$$

By differentiating $\tilde{\beta}$ with respect to n we obtain:

$$\frac{\partial\tilde{\beta}}{\partial n} = -\frac{\delta\,(1-\lambda)^2}{(2n+\delta\Lambda)^2}.$$

Thus, for $\lambda < 1$ and convex demand ($\delta < 0$), $\partial\tilde{\beta}/\partial n > 0$; if demand is concave ($\delta > 0$), $\partial\tilde{\beta}/\partial n < 0$. Let us now differentiate $\tilde{\beta}$ with respect to λ:

$$\frac{\partial\tilde{\beta}}{\partial\lambda} = \frac{n^2(\delta+2)}{(2n+\delta\Lambda)^2},$$

then, $\partial\tilde{\beta}/\partial\lambda > 0$ if $\delta > -2$. Finally, we differentiate $\tilde{\beta}$ with respect to δ:

$$\frac{\partial\tilde{\beta}}{\partial\delta} = \frac{\Lambda n\,(1-\lambda)}{(2n+\delta\Lambda)^2}.$$

Thus, $\partial\tilde{\beta}/\partial\delta > 0$ if $\lambda < 1$.■

Proof of Lemma 4. Using (12), by totally differentiating the system formed by (10; 11) in a symmetric equilibrium, and solving for $\partial q^*/\partial\lambda$ and $\partial x^*/\partial\lambda$, we obtain

$$\frac{\partial q^*}{\partial\lambda} = \frac{1}{\tilde{\Delta}}\left\{\left[\partial_{\lambda x_i}\phi_i + (n-1)\psi_\lambda\right]\left(\partial_{x_i q_i}\phi_i\right)B - \partial_{\lambda q_i}\phi_i\left[\Delta_x + \psi_x(n-1)\right]\right\} \tag{39}$$

$$\frac{\partial x^*}{\partial\lambda} = \frac{1}{\tilde{\Delta}}\left\{\partial_{\lambda q_i}\phi_i\left[\partial_{x_i q_i}\phi_i\tau + (n-1)\psi_q\right] - \left[\partial_{\lambda x_i}\phi_i + (n-1)\psi_\lambda\right]\Delta_q\right\}, \tag{40}$$

where $\psi_z \equiv \partial\psi/\partial z$ with $z = q, x, \lambda$, and

$$\tilde{\Delta}(Q^*, x^*) = \Delta_q\left[\Delta_x + \psi_x(n-1)\right] - \partial_{x_i q_i}\phi_i\left[\partial_{x_i q_i}\phi_i\tau + \psi_q(n-1)\right]B,$$

which is assumed to be strictly positive.[4] By rewriting equation (40) as follows

$$\frac{\partial x^*}{\partial\lambda} = \vartheta f'(Q^*)c'(Bx^*)\left\{(\beta + s'(\lambda))\left[\Lambda(1+\delta) + n\right] - \left[\tau + (n-1)s(\lambda)\right]\right\}, \tag{41}$$

where $\vartheta \equiv (n-1)(Q^*/n)/\tilde{\Delta}$ and $s(\lambda) \equiv \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)$, we get that sign$\{\partial x^*/\partial\lambda\}$ is given by (15). Let us now turn to the impact of λ on output in equilibrium. Equation (39) can be rewritten as follows

$$\frac{\partial q^*}{\partial\lambda} = \vartheta\left((\beta + s'(\lambda))c'(Bx^*)^2 B + f'(Q^*)\left\{c''(Bx^*)(Q^*/n)B\left[\tau + (n-1)s(\lambda)\right] + \Gamma''(x^*)\right\}\right). \tag{42}$$

By inserting the FOC (11) evaluated in the symmetric equilibrium into the above expression, after some manipulations we get that sign$\{\partial q^*/\partial\lambda\}$ is given by (16). Finally, note that the FOC with respect to output is identical to the one associated to the static case. Therefore, we obtain again equation (7), which implies that if $\partial x^*/\partial\lambda \leq 0$, then $\partial q^*/\partial\lambda < 0$. From (15), we obtain that $\partial x^*/\partial\lambda > 0$ if and only if

$$\beta > \underline{\beta}^{2S} \equiv \frac{1 - (\omega'(\lambda)\tilde{\beta}(\lambda) + \omega(\lambda)\tilde{\beta}'(\lambda))P'(c)^{-1}n + \omega(\lambda)(n-1)\tilde{\beta}(\lambda)}{(1 + n + \Lambda\delta) + (n-1)\omega(\lambda) - P'(c)^{-1}n\omega'(\lambda)}. \blacksquare$$

LEMMA 7 *Under assumptions A.1.-A.4, in the two-stage model, there is a cut-off spillover value for spillovers ($\bar{\beta}^{2S} < 1$) above which allowing some overlapping ownership is socially optimal ($\lambda^o_{\mathrm{TS}} > 0$) if*

$$(1 + s'(0))n + (1 - s(0))(n-1)((1 + s'(0))(1 + \delta + n) - [1 + (n-1)s(0)] - H(1) > 0. \tag{43}$$

[4] We show in Section A.2.2 that $\tilde{\Delta}(Q^*, x^*) > 0$ is also a necessary condition for having a positive output at equilibrium in AJ.

Proof. By differentiating $W(\lambda)$ we have

$$W'(\lambda) = [f(Q^*) - c(Bx^*)]n\frac{\partial q^*}{\partial \lambda} - c'(Bx^*)BQ^*\frac{\partial x^*}{\partial \lambda} - n\Gamma'(x^*)\frac{\partial x^*}{\partial \lambda}.$$

Using the FOCs, $f(Q^*) - c(Bx^*) = -f'(Q^*)Q^*\Lambda/n$ and (14) in the above expression, and simplifying, we obtain:

$$W'(\lambda) = \left\{-\Lambda f'(Q^*)\frac{\partial q^*}{\partial \lambda} - [(1-\lambda)\beta - s(\lambda)]\,(n-1)c'(Bx^*)\frac{\partial x^*}{\partial \lambda}\right\}Q^*. \tag{44}$$

If we insert (41) and (42) into (44), after some manipulations we get

$$\begin{aligned} W'(\lambda) \;=\;& \vartheta_w Q^*(-f'(Q^*))\left[\Lambda\left(c'(Bx^*)^2(\beta + s'(\lambda))B\right.\right. \\ &+ f'(Q^*)\left\{c''(Bx^*)(Q^*/n)B\left[\tau + (n-1)s(\lambda)\right] + \Gamma''(x^*)\right\}\big) \\ &+ c'(Bx^*)^2\left[(1-\lambda)\beta - s(\lambda)\right](n-1)\left\{(\beta + s'(\lambda))\left[\Lambda(1+\delta) + n\right]\right. \\ &\left.\left. - \left[\tau + (n-1)s(\lambda)\right]\right\}\right], \end{aligned} \tag{45}$$

where $\vartheta_w \equiv (n-1)(Q^*/n)/\tilde{\Delta}$. Then $W'(0)|_{\beta=1} > 0$ if and only if

$$\begin{aligned} 0 \;<\;& (c'(nx^*))^2\left((1 + s'(0)|_{\beta=1})n + (1 - s(0)|_{\beta=1})(n-1)\left\{(1 + s'(0)|_{\beta=1})(1+\delta+n)\right.\right. \\ &\left.\left. - \left[1 + (n-1)\;s(0)|_{\beta=1}\right]\right\}\right) + f'(Q^*)\left\{c''(nx^*)Q^*\left[1 + (n-1)\;s(0)|_{\beta=1}\right] + \Gamma''(x^*)\right\}. \end{aligned} \tag{46}$$

From equation (14) we have that in equilibrium and for $\lambda = 0$ and $\beta = 1$:

$$Q^*|_{\lambda=0,\beta=1} = -\frac{n\Gamma'(x^*)}{c'(nx^*)\left[1 + (n-1)\;s(0)|_{\beta=1}\right]}.$$

Substituting $Q^*|_{\lambda=0,\beta=1}$ into (46) and using the definitions for $\chi(Bx^*)$ and $\xi(Q^*, x^*)$, we obtain the condition for the two-period model:

$$\begin{aligned} 0 \;<\;& (1 + s'(0)|_{\beta=1})n + (1 - s(0)|_{\beta=1})(n-1)\left\{(1 + s'(0)|_{\beta=1})(1+\delta+n)\right. \\ &\left. - \left[1 + (n-1)\;s(0)|_{\beta=1}\right]\right\} - H(1), \end{aligned}$$

where

$$s(0) = \frac{(2n+\delta)[(n+\delta)/(2n+\delta) - \beta]}{n(n+1+\delta)}$$

Table 2: Model Specifications

	AJ	KMZ	CE
Demand	$f(Q) = a - bQ$	$f(Q) = a - bQ$	$f(Q) = \sigma Q^{-\varepsilon}$, $0 < \varepsilon < 1$
	$\delta = 0$; $a, b > 0$	$\delta = 0$; $a, b > 0$	$\delta = -(1+\varepsilon)$; $a = 0, b = -\sigma < 0$
$c(\cdot)$	$\bar{c} - x_i - \beta \sum_{j \neq i} x_j$	$\bar{c} - [(2/\gamma)(x_i + \beta \sum_{j \neq i} x_j)]^{1/2}$	$\kappa(x_i + \beta \sum_{j \neq i} x_j)^{-\alpha}$; $\alpha, \kappa > 0$
$\Gamma(x)$	$(\gamma/2)x^2$	x	x

and

$$s'(0) = -\frac{\left[2n^2 + \delta(2n+1) + \delta^2\right](n-1)\beta - \delta^2(n-1) - \delta(2n^2-1) - n(n^2+1)}{(n+1+\delta)^2 n}.$$

Thus, $s'(0)|_{\beta=1} = \left[1 + \delta - n(n-2)\right]/(n+1+\delta)^2$. Note that by setting $s = s' = 0$, we obtain the condition for the simultaneous case, that is, (31).■

A.2 The three model specifications

In this section we characterize each of the model specifications considered in the paper: first in the simultaneous and then in the two-stage model. First we describe briefly the main assumptions of each model specification.

As shown in Amir (2000) the AJ and the KMZ model specifications are not equivalent for large spillover values (the critical value depends on the innovation function and on the number of firms). The difference between the two models lies on the innovation function and the autonomous R&D expenditures. Under the KMZ specification, *the effective R&D investment* for each firm is the sum of its own expenditure x_i and a fixed fraction (β) of the sum of the expenditures of the rest of firms, i.e., $X_i = x_i + \beta \sum_{j \neq i} x_j$. Instead, under the AJ specification, X_i is *the effective cost reduction* for each firm, so $c(\cdot)$ is a linear function. Thus, in AJ decision variables are unit-cost reductions, whereas in KMZ decision variables are the autonomous R&D expenditures. In particular, in KMZ the unit cost of firm i is $\bar{c} - h(x_i + \beta \sum_{j \neq i} x_j)$, where for given $x_i \geq 0$ $(i = 1, ..., n)$ the effective cost reduction to firm i, $h(\cdot)$, is a twice differentiable and concave function with $h(0) = 0$, $h(\cdot) \leq \bar{c}$, and $(\partial/\partial x_i)h(\cdot) > 0$. As in Amir (2000), to allow for a direct comparison between AJ and KMZ, we consider a particular case of the KMZ model: $h = [(2/\gamma)(x_i + \beta \sum_{j \neq i} x_j)]^{1/2}$ with $\gamma > 0$. The CE model considers constant elasticity demand and costs with $\alpha, \kappa > 0$ (see Table 2); α is the unit cost of production (or innovation function) elasticity with respect to the investment in R&D and there are no spillover effects. Note that the assumption $\varepsilon < 1$ implies that $\delta > -2$, and consequently quantities are *strategic substitutes.*

Finally, $\Gamma(x)$ is quadratic in AJ but linear in KMZ and CE.

A.2.1 Simultaneous model

We first discuss comparative statics on equilibrium values given in Table A1, and then derive Table A2, which provides the second-order and regularity conditions for the three model specifications (we also explore the feasible region for the constant elasticity model in Lemma A1). Second, we establish Lemma A2, which determines sign$\{\partial q^*/\partial\lambda\}$ and sign$\{\partial x^*/\partial\lambda\}$ for each model specification. Third, we derive the spillover threshold value $\bar{\beta}$ and $\beta'(0)$ in the examples (Table A3). After that, we conduct a comparative statics analysis on $\bar{\beta}$. Finally, we examine welfare in AJ and KMZ, obtain the optimal degree of overlapping ownership in each case (Table A4) and state and prove Proposition A1.

Table A1: Equilibrium Values

	AJ	KMZ	CE
q^*	$\frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n)-B\tau}$	$\frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n)-\tau}$	$\frac{1}{\alpha\kappa\tau}\left[\sigma\left(\tau\alpha/n\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon\Lambda/n\right)\right]^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$
x^*	$\frac{\tau(a-\bar{c})}{\gamma b(\Lambda+n)-B\tau}$	$\frac{\gamma\tau^2(a-\bar{c})^2}{2B[\gamma b(\Lambda+n)-\tau]^2}$	$\frac{1}{B}\left[\sigma\left(\tau\alpha/n\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon\Lambda/n\right)\right]^{1/[\varepsilon-\alpha(1-\varepsilon)]}$

Table A2: Second-Order Conditions and Regularity Condition

	AJ	KMZ	CE
S.O.C	$\gamma b > 1/2$	$\gamma b > \tau/(2\tilde{\lambda})$	$n > \frac{\Lambda(1+\varepsilon)}{2}$ *and* $\frac{\varepsilon(1+\alpha)}{\alpha} > \frac{n(n-\varepsilon\Lambda)}{\tilde{\lambda}(2n+\Lambda\delta)}$
Regularity Condition	$\gamma b > \tau B/(\Lambda+n)$	$\gamma b > \tau/(\Lambda+n)$	$\varepsilon-\alpha(1-\varepsilon) > 0$

with $\tilde{\lambda} \equiv 1+\lambda(n-1)\beta^2$.

Table A3: Spillover Thresholds $\bar{\beta}$ and $\beta'(0)$

	$\bar{\beta}$	$\beta'(0)$
AJ	$\frac{(n-2)+\sqrt{(n-2)^2+4b\gamma(n+2)(n-1)}}{2(n+2)(n-1)}$	$[-1+\sqrt{1+4b\gamma(n-1)}]/[2(n-1)]$
KMZ	$\frac{(n-2)+b\gamma(n-1)+\sqrt{(n-2)^2+b\gamma(n-1)\left[b\gamma(n-1)+6n+4\right]}}{2(n+2)(n-1)}$	γb
CE	is the value above which: $(n-\varepsilon)\alpha\beta\left\{B+(n-1)\left[\beta(n-\varepsilon)-1\right]\right\}-\varepsilon(\alpha+1)B > 0$	$\varepsilon(\alpha+1)/[\alpha(n-\varepsilon)]$

Comparative statics on equilibrium values. In AJ and KMZ the R&D expenditure x^* and output q^* per firm increase with the size of the market (a) and decrease with the level of inefficiency of the technology employed, $\bar{c}$, the slope of inverse demand, b, and the parameter

γ (which is the parameter of the slope of the R&D costs in AJ). In the CE model x^* and q^* increase with the size of the market, σ. In addition, the costlier the technology employed, κ, the lower is total output, Q^*. However, x^* decreases (respectively, increases) with κ if demand is elastic (inelastic). The last two results hold for any value of λ and β.[5]

Derivation of Table A2. In AJ and KMZ it is immediate that $\partial_{q_i q_i}\phi_i = -2b < 0$. Furthermore, in AJ: condition $\partial_{q_i q_i}\phi_i\left(\partial_{x_i x_i}\phi_i\right) - \left(\partial_{q_i x_i}\phi_i\right)^2 > 0$, given by (21), can be written as $2b\gamma - 1 > 0$, since $c''(\cdot) = 0$ and $\Gamma''(x) = \gamma$, so $\partial_{x_i x_i}\phi_i = -\gamma$ and $\partial_{q_i x_i}\phi_i = -c'(\cdot) = 1$. In KMZ, (21) can be written as

$$\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}(Bx^*)\right)^{-1}\right] - 2b\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}(Bx^*)\right)^{-3/2}\right] q^*\tilde{\lambda} < 0. \tag{47}$$

From FOC (3) we have that in equilibrium

$$q^* = \frac{\Gamma'(x^*)}{-c'(Bx^*)\tau} = \frac{1}{(1/\gamma)\left[(2(Bx^*)/\gamma\right]^{-1/2}\tau}. \tag{48}$$

Inserting the above equation into condition (47), after some manipulations, it reduces to $1 - 2b\gamma\tilde{\lambda}/\tau < 0$. (Note that if $\gamma b > \tau/2$ holds, then the condition $\gamma b > \tau/(2\tilde{\lambda})$ is satisfied.) In AJ and from (20), it is immediate that $\Delta = \gamma b(\Lambda + n) - \tau B$ since $c''(\cdot) = \delta = 0$, $f'(Q) = -b$ and $\Gamma'(x) = \gamma x$. In KMZ we have:

$$\begin{aligned}
\Delta &= -\left[\frac{1}{\gamma^2}\left(\frac{2}{\gamma}Bx^*\right)^{-3/2} B\tau\frac{1}{(1/\gamma)(2Bx^*/\gamma)^{-1/2}\tau}\right]\left[-b\left(\Lambda + n\right)\right] - \frac{1}{\gamma^2}\left(\frac{2}{\gamma}Bx^*\right)^{-1}\tau B \\
&= \frac{1}{\gamma}\left(\frac{2}{\gamma}Bx^*\right)^{-1}\left[Bb(\Lambda + n) - \frac{\tau B}{\gamma}\right].
\end{aligned}$$

Therefore, in KMZ $\Delta > 0$ if $\gamma b > \tau/(\Lambda + n)$. Regarding the constant elasticity model we have:

LEMMA A1 *(Constant elasticity model) At the equilibrium, for a given $n \geq 2$ and $\lambda \geq 0$, second order conditions together with the condition of non-negative profits require that*

(i) $\max\{\varepsilon\Lambda, \Lambda(1+\varepsilon)/2\} < n \leq \varepsilon\Lambda(B + \alpha\tau)/(\alpha\tau)$,

(ii) $\varepsilon(1+\alpha)/\alpha > n(n - \varepsilon\Lambda)/\left[\tilde{\lambda}(2n + \Lambda\delta)\right]$, *with* $\tilde{\lambda} \equiv 1 + \lambda(n-1)\beta^2$.

Furthermore, the equilibrium is regular if and only if $(1+\alpha)/\alpha > 1/\varepsilon$.

Proof. From the FOC (2) we need that

$$n > \varepsilon\Lambda, \tag{49}$$

[5] The same result is obtained in Dasgupta and Stiglitz (1980) for $\lambda = \beta = 0$ and free entry.

otherwise the system (2; 3) will not have a solution. Since $\delta = -(1+\varepsilon)$, $\Delta_q < 0$ if condition (49) holds (see Table 4). This condition also guarantees that Q^* and x^* are both positive. Notice that $\partial_{q_i q_i}\phi_i < 0$ if $(f'(Q^*)/n)(2n + \Lambda\delta) < 0$, then $\partial_{q_i q_i}\phi_i < 0$ if

$$n > \Lambda(1+\varepsilon)/2. \tag{50}$$

Since $\Lambda \in [1, n]$, we have that the latter condition is always satisfied for $\varepsilon < 1$. By construction $\partial_{x_i x_i}\phi_i < 0$. Furthermore, second order condition $\partial_{q_i q_i}\phi_i\left(\partial_{x_i x_i}\phi_i\right) - \left(\partial_{q_i x_i}\phi_i\right)^2 > 0$, which is given by (21), reduces to

$$-\frac{\varepsilon\sigma}{n}Q^{*-(\varepsilon+1)}(2n+\Lambda\delta)\left[\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tilde{\lambda}\right] + (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)} < 0. \tag{51}$$

From the FOC (2) we have that at the symmetric equilibrium

$$Q^* = \left[\sigma(n-\varepsilon\Lambda)/(n\kappa)\right]^{1/\varepsilon}(Bx^*)^{\alpha/\varepsilon}. \tag{52}$$

By substituting (52) into (51), after some manipulations, we obtain

$$(Bx^*)^{-2(\alpha+1)}\alpha\kappa^2\left\{-\left[\varepsilon/(n-\varepsilon\Lambda)\right](2n+\Lambda\delta)(\alpha+1)\tilde{\lambda}/n + \alpha\right\} < 0.$$

The above condition is satisfied if $\varepsilon(\alpha+1)/\alpha > n(n-\varepsilon\Lambda)/[(2n+\Lambda\delta)\tilde{\lambda}]$, which proves statement (ii) of the Proposition.

From (20) we have that $\Delta > 0$ if

$$\begin{aligned} 0 \;&<\; -\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tau B\left[\varepsilon(1+\varepsilon)\sigma Q^{*-(\varepsilon+2)}\Lambda Q^* - \varepsilon\sigma Q^{*-(\varepsilon+1)}(\Lambda+n)\right] \\ &\quad -(\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B, \text{ or} \end{aligned}$$

$$\begin{aligned} 0 \;&<\; Q^{*-(\varepsilon+1)}\left[-\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}(Q^*/n)\tau B\right]\left[\varepsilon(1+\varepsilon)\sigma\Lambda - \varepsilon\sigma(\Lambda+n)\right] \\ &\quad -(\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B. \end{aligned}$$

Substituting (52) into the above expression, we obtain

$$\begin{aligned} 0 \;&<\; -\left[\frac{\sigma(n-\varepsilon\Lambda)}{n\kappa}\right]^{-(\varepsilon+1)/\varepsilon}(Bx^*)^{-(\varepsilon+1)\alpha/\varepsilon}\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}\left[\frac{\sigma(n-\varepsilon\Lambda)}{n\kappa}\right]^{1/\varepsilon}(Bx^*)^{\alpha/\varepsilon} \\ &\quad \frac{\tau B}{n}\left[\varepsilon(1+\varepsilon)\sigma\Lambda - \varepsilon\sigma(\Lambda+n)\right] - (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\tau B, \end{aligned}$$

rearranging terms yields

$$0 < (Bx^*)^{-2(\alpha+1)} \left[\frac{n\kappa}{\sigma(n-\varepsilon\Lambda)} \left(-\alpha(\alpha+1)\frac{\kappa\tau B}{n} \right) \left(-\varepsilon\sigma n + \varepsilon^2\sigma\Lambda \right) - (\alpha\kappa)^2\tau B \right], \text{ or equivalently,}$$

$$0 < (Bx^*)^{-2(\alpha+1)}\alpha\kappa^2\tau B \left[\varepsilon(\alpha+1) - \alpha \right].$$

Therefore, $\Delta > 0$ holds if $(1+\alpha)/\alpha > 1/\varepsilon$, or, equivalently, if $\varepsilon - \alpha(1-\varepsilon) > 0$.

We turn now to deriving the condition under which profits in equilibrium are nonnegative. At the symmetric equilibrium, each firm's profit is given by $\pi(Q^*/n, x^*) = [f(Q^*) - c(Bx^*)]\,(Q^*/n) - x^*$. Then, $\pi(Q^*/n, x^*) \geq 0$ if $\bar{\pi} \equiv [f(Q^*) - c(Bx^*)]\,Q^*/(x^*n) \geq 1$. Write

$$\vartheta_{CE} \equiv \sigma \left(\frac{\tau\alpha}{n} \right)^{\varepsilon} \kappa^{\varepsilon-1} \left(\frac{n-\varepsilon\Lambda}{n} \right).$$

Then $Q^* = [n/(\alpha\kappa\tau)]\,{\vartheta_{CE}}^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$, $x^* = (1/B)\,{\vartheta_{CE}}^{1/[\varepsilon-\alpha(1-\varepsilon)]}$, and condition $\bar{\pi} \geq 1$ can be expressed as

$$\left[\sigma \left(\frac{n}{\alpha\kappa\tau} \right)^{-\varepsilon} {\vartheta_{CE}}^{-\varepsilon(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]} - \kappa\, {\vartheta_{CE}}^{-\alpha/[\varepsilon-\alpha(1-\varepsilon)]} \right] \frac{1}{\alpha\kappa\tau}\, {\vartheta_{CE}}^{\alpha/[\varepsilon-\alpha(1-\varepsilon)]}\, B \geq 1.$$

Rearranging terms, and replacing ϑ_{CE} into the above expression, we get $[\varepsilon\Lambda/(n-\varepsilon\Lambda)]\,[B/(\alpha\tau)] \geq 1$. It follows that $\bar{\pi} \geq 1$ if

$$\left(\frac{\varepsilon\Lambda}{\alpha\tau} \right) (B + \alpha\tau) \geq n. \tag{53}$$

Combining conditions (49), (50) and (53) yields statement (i).■

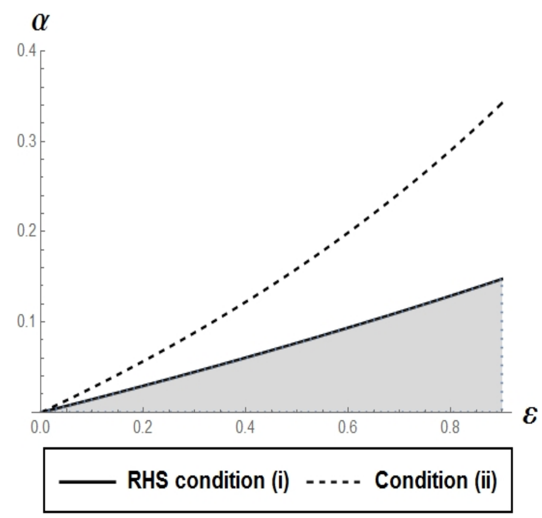

Fig. A1. Feasible region for the CE model with $n = 7$.

Feasible region for the constant elasticity model with $\lambda = 0$. From Lemma A1 we have that $\Delta > 0$ if $(1+\alpha)/\alpha > 1/\varepsilon$. When $\lambda = 0$, the LHS of condition (i) is satisfied for any $n \geq 2$ since $\varepsilon < 1$, moreover the RHS of condition (i) can be rewritten as follows $n \leq \rho_{CE}(\beta) = \varepsilon(1+\alpha-\beta)/(\alpha-\varepsilon\beta)$. Since $\rho'_{CE} > 0$ (as we are also imposing that $\Delta > 0$), condition $n \leq \rho_{CE}(\beta)$ will hold for all β if $n \leq \varepsilon(1+\alpha)/\alpha$. Last, condition (ii) with $\lambda = 0$ writes as $\varepsilon(1+\alpha)/\alpha > n(n-\varepsilon)/\left[2n-(1+\varepsilon)\right]$. Therefore, at $\lambda = 0$ we only have to consider the RHS of condition (i) and condition (ii). These two conditions are depicted in Fig. A1 for $n = 7$; the grey area are combinations (α, ε) for which the two conditions are satisfied (these combinations of parameters also satisfy the two conditions for $n \leq 7$).

Determination of $\text{sign}\{\partial q^*/\partial\lambda\}$ and $\text{sign}\{\partial x^*/\partial\lambda\}$ in AJ, KMZ and CE. Note that $\partial q^*/\partial\lambda$ can be written in the following manner

$$\frac{\partial q^*}{\partial \lambda} = \frac{(n-1)(Q^*/n)}{\Delta}\left\{\left(c'(Bx^*)\right)^2 \beta B + f'(Q^*)\left[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)\right]\right\}, \tag{54}$$

then after some calculations, it is simple to verify that in the simultaneous model:

LEMMA A2 *We have (i) In AJ:* $\text{sign}\left\{\dfrac{\partial q^*}{\partial\lambda}\right\} = \text{sign}\{\beta(1+\beta(n-1))-b\gamma\}$ *and* $\text{sign}\left\{\dfrac{\partial x^*}{\partial\lambda}\right\} = \text{sign}\{\beta(n+1)-1\}$*; (ii) In KMZ:* $\text{sign}\left\{\dfrac{\partial q^*}{\partial\lambda}\right\} = \text{sign}\{\beta-\gamma b\}$ *and* $\text{sign}\left\{\dfrac{\partial x^*}{\partial\lambda}\right\} = \text{sign}\{\beta(n+1)-1\}$*; (iii) In the CE model:* $\text{sign}\left\{\dfrac{\partial q^*}{\partial\lambda}\right\} = \text{sign}\{\beta\left[\alpha(n-\varepsilon\Lambda)-\lambda(n-1)\varepsilon(\alpha+1)\right]-\varepsilon(\alpha+1)\}$

and sign $\left\{\dfrac{\partial x^*}{\partial \lambda}\right\} = \text{sign}\{\beta\left[(n-\varepsilon) - \lambda(n-1)(1+\varepsilon)\right] - 1\}.$

Derivation of $\bar{\beta}$ (Table A3). Note that $\partial x^*/\partial\lambda$ can be written as

$$\frac{\partial x^*}{\partial \lambda} = \frac{(n-1)(Q^*/n)f'(Q^*)c'(Bx^*)}{\Delta}[\beta(\Lambda(1+\delta)+n)-\tau] \tag{55}$$

If we insert equations (54) and (55) into equation (9), after some manipulations we obtain $W'(\lambda) = \left[(n-1)(Q^*)^2/(n\Delta)\right](-f'(Q^*))\digamma$, where

$$\begin{aligned}\digamma &\equiv \Lambda\{\left(c'(Bx^*)\right)^2\beta B + f'(Q^*)[c''(Bx^*)(Q^*/n)B\tau + \Gamma''(x^*)]\} \\ &\quad + \left(c'(Bx^*)\right)^2(1-\lambda)\beta(n-1)\left\{\beta\left[\Lambda(1+\delta)+n\right]-\tau\right\}.\end{aligned}$$

By noting that in AJ: $f' = -b$, $\delta = 0$, $c' = -1$, $c'' = 0$ and $\Gamma'' = \gamma$, it then follows that

$$\begin{aligned}\digamma^{AJ} &= \digamma|_{\lambda=0} = \beta B - b\gamma + \beta(n-1)\left[\beta(1+n)-1\right] \\ &= (n-1)(n+2)\beta^2 - (n-2)\beta - b\gamma.\end{aligned}$$

By solving $\digamma^{AJ} = 0$ for β we obtain the expression for $\bar{\beta}^{AJ}$. Notice that $\bar{\beta}^{AJ} < 1$ if

$$(n-2) + \sqrt{(n-2)^2 + 4b\gamma(n+2)(n-1)} < 2(n+2)(n-1),$$

or

$$(n-2)^2 + 4b\gamma(n+2)(n-1) < \left[2(n+2)(n-1)-(n-2)\right]^2,$$

which can be rewritten as $4b\gamma(n+2)(n-1) < 4n^2(n+2)(n-1)$. Thus, $\bar{\beta}^{AJ} < 1$ if $b\gamma < n^2$. In KMZ we have $c = \bar{c} - \sqrt{(2/\gamma)(x_i + \beta\sum_{j\neq i} x_j)}$, $f' = -b$, $\delta = 0$ and $\Gamma'' = 0$, then

$$\begin{aligned}\digamma^{KMZ} &= \digamma|_{\lambda=0} = \frac{\beta}{2\gamma x^*} + \frac{-bq^*B}{\gamma^2\left(2Bx^*/\gamma\right)^{3/2}} + \frac{\beta(n-1)\left[\beta(1+n)-1\right]}{2\gamma Bx^*} \\ &= \frac{1}{B}\left(\frac{-bq^*B^{1/2}}{\gamma^2\left(2x^*/\gamma\right)^{3/2}} + \frac{\beta}{2\gamma x^*}\left\{B + (n-1)\left[\beta(1+n)-1\right]\right\}\right).\end{aligned}$$

By replacing q^* and x^* into the above expression, after some calculations we get

$$\digamma^{KMZ} = \frac{\left[b\gamma(1+n)-1\right]^2}{\gamma(a-\bar{c})^2}\left(-bB + \frac{\beta}{\gamma}\left\{B + (n-1)\left[\beta(1+n)-1\right]\right\}\right).$$

It is then immediate that: $\digamma^{KMZ} > 0 \Leftrightarrow \beta > \bar{\beta}^{KMZ}$. Notice that $\bar{\beta}^{KMZ} < 1$ if

$$\left\{(n-2)^2 + b\gamma(n-1)\left[b\gamma(n-1) + 2(3n+2)\right]\right\}^{1/2} < 2(n+2)(n-1) - n + 2 - b\gamma(n-1),$$

which can be rewritten as $4n(n+2)(n-1)(-n+b\gamma) < 0$. In the constant elasticity model $f = \sigma Q^{-\varepsilon}$, $c = \kappa(x_i + \beta\sum_{j\neq i} x_j)^{-\alpha}$ and $\Gamma(x) = x$, then

$$\begin{aligned} \digamma^{CE} &= \digamma|_{\lambda=0} = (\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\beta B - \varepsilon\sigma(Q^*)^{-\varepsilon-1}\alpha(\alpha+1)\kappa(Bx^*)^{-(\alpha+2)}q^*B \\ &\quad +(\alpha\kappa)^2(Bx^*)^{-2(\alpha+1)}\beta(n-1)\left[\beta(-\varepsilon+n)-1\right]. \end{aligned}$$

By replacing q^* and x^* into the above expression, we obtain

$$\begin{aligned} \digamma^{CE} &= \alpha^2\kappa^2 z^{-2(1+\alpha)}\beta B - \varepsilon\sigma\left[n/(\alpha\kappa)\right]^{-(1+\varepsilon)} z^{-(1+\alpha)(1+\varepsilon)}(\alpha+1)z^{-(\alpha+2)}z^{\alpha+1}B \\ &\quad +\alpha^2\kappa^2 z^{-2(1+\alpha)}\beta(n-1)\left[\beta(-\varepsilon+n)-1\right], \end{aligned} \tag{56}$$

where

$$z \equiv \left[\sigma\left(\frac{\tau\alpha}{n}\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\varepsilon/n\right)\right]^{1/[\varepsilon-\alpha(1-\varepsilon)]}.$$

By noting that $z^{-(\alpha+1)(1+\varepsilon)-(\alpha+2)+(\alpha+1)} = z^{-\varepsilon+\alpha(1-\varepsilon)}z^{-2(1+\alpha)}$ we can re-write equation (56) as follows

$$\digamma^{CE} = z^{-2(1+\alpha)}\alpha\kappa^2\left\{\alpha\beta B + \alpha\beta(n-1)\left[\beta(-\varepsilon+n)-1\right] - \varepsilon(\alpha+1)B/(n-\varepsilon)\right\}.$$

Hence $\digamma^{CE} > 0$ if and only if

$$(n-\varepsilon)\alpha\beta\left\{B + (n-1)\left[\beta(n-\varepsilon)-1\right]\right\} - \varepsilon(\alpha+1)B > 0.\blacksquare$$

Threshold value $\bar{\beta}$

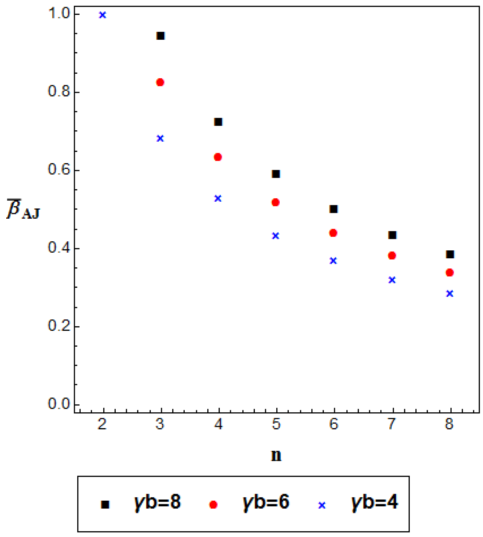

Fig. A2a. AJ model.

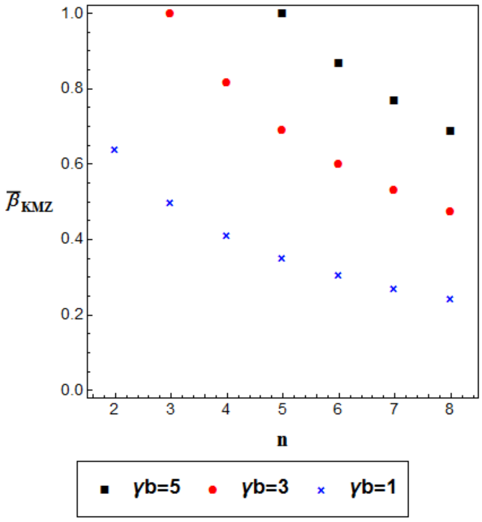

Fig. A2b. KMZ model.

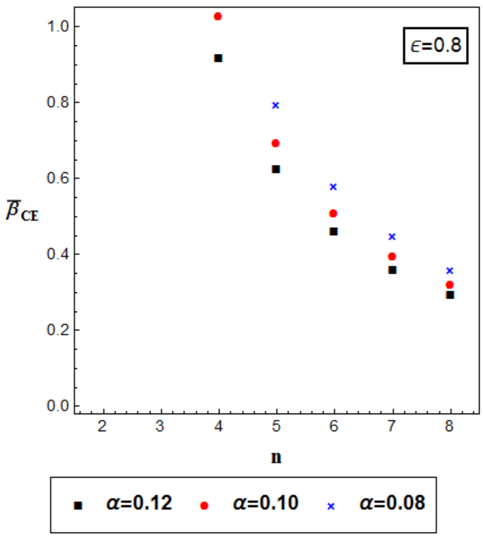

Figure A3a. CE model.

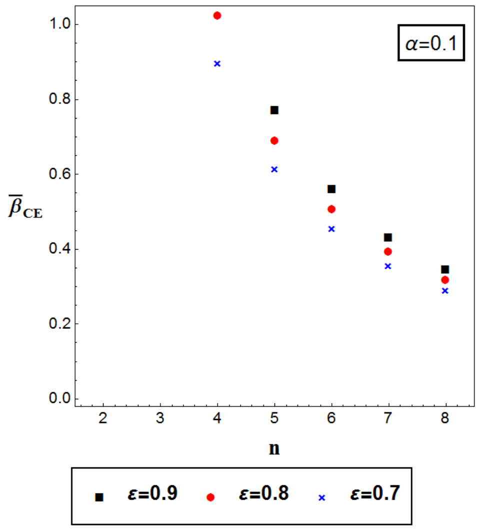

Figure A3b. CE model.

Comparative statics on $\bar{\beta}$. Fig. A2a (respectively Fig. A2b) shows the value for $\bar{\beta}$ under the AJ (KMZ) model specification as a function of the number of firms and for different values of γb. As the figure makes clear, $\bar{\beta}^{AJ}$ and $\bar{\beta}^{KMZ}$ decrease with n: when there are more firms in the market, there is more need for overlapping ownership in order to internalize the additional externalities. We also have that $\bar{\beta}^{AJ}$ and $\bar{\beta}^{KMZ}$ decrease with γb, although $\bar{\beta}$ is lower than 1 for lower values of γb in the KMZ model than in the AJ model.

Fig. A3a and Fig. A3b depict $\bar{\beta}^{CE}$ as a function of n and for different values for α and ε. A glance at these figures shows that $\bar{\beta}^{CE}$ decreases again with n (for given ε and α). In

Table A4: Optimal Degree of Cross-ownership in AJ and KMZ

	λ^o_{TS}
AJ	$\min\left\{\max\left\{0, \frac{[(n+2)(n-1)\beta-(n-2)]\beta-b\gamma}{(n-1)[2(\beta-1)\beta+b\gamma]}\right\}, 1\right\}$
KMZ	$\min\left\{\max\left\{0, \frac{[(n+2)(n-1)\beta-b\gamma(n-1)-(n-2)]\beta-b\gamma}{(n-1)\{[2\beta+b\gamma(n-1)-2]\beta+b\gamma\}}\right\}, 1\right\}$

addition, Fig. A3a tells us that for given n and ε, $\bar{\beta}^{CE}$ decreases with the elasticity of the innovation function, α, whereas Fig. A3b shows that for given n and α, $\bar{\beta}^{CE}$ increases with ε, so it decreases with the elasticity of demand. We also have that for the (feasible) combination of parameters (α, ε) considered here, $\bar{\beta}^{CE} \geq 1$ when there are two or three firms in the market.

Optimal degree of overlapping ownership (TS and CS standard)

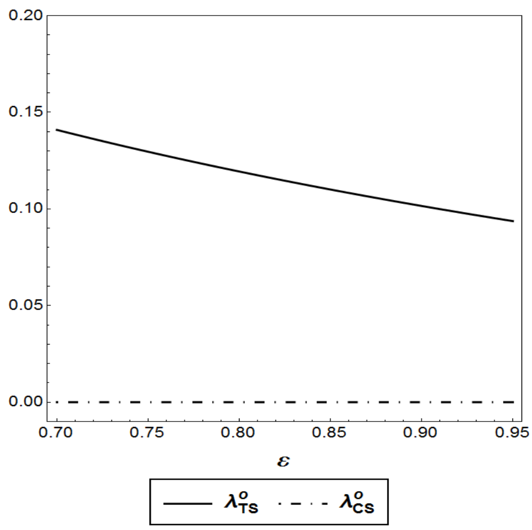

Fig. A4a. CE model.

($\alpha = 0.1$, $\sigma = \kappa = 1$, $n = 8$, $\beta = 0.8$)

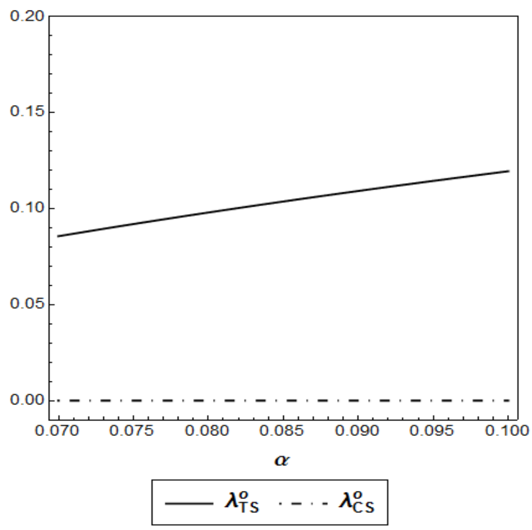

Fig. A4b. CE model.

($\varepsilon = 0.8$, $\sigma = \kappa = 1$, $n = 8$, $\beta = 0.8$)

Fig. A4a and A4b show that the greater is the elasticity of demand, ε^{-1}, or the elasticity of the innovation function, α, the greater should be the degree of overlapping ownership if the social planner seeks to maximize total surplus; however, if the objective is to maximize consumer surplus, then for the same parameter range, $\lambda^o_{\mathrm{CS}} = 0$.

Welfare in AJ and KMZ. Here, we show that welfare is a single-peaked function in AJ and KMZ; we also derive λ^o_{TS} under these two model specifications (Table A4).

Case AJ: By inserting equilibrium values into the welfare function we get

$$W = \frac{1}{2}n\gamma(a-\bar{c})^2\frac{(2\Lambda+n)\gamma b-\tau^2}{[(\Lambda+n)\gamma b-B\tau]^2}.$$

If we differentiate W with respect to λ we obtain:

$$\frac{dW}{d\lambda} = -\frac{(n-1)(a-\bar{c})\gamma b\left\{\Lambda\gamma b+\beta\left[2\lambda\left(B-n\right)+n-2-\beta(n+2)(n-1)\right]\right\}}{\left[\left(\Lambda+n\right)b\gamma-B\tau\right]^2}Q.$$

Note that solving $dW/d\lambda = 0$ for λ yields a unique stationary point, given by $\hat{\lambda}_{AJ}$. By taking the second order derivative with respect to λ, evaluating it at $\lambda = \hat{\lambda}_{AJ}$, and simplifying, we obtain

$$\left.\frac{d^2W}{d\lambda^2}\right|_{\lambda=\hat{\lambda}_{AJ}} = -\frac{(n-1)^2(a-\bar{c})\gamma b\left[2\left(\beta-1\right)\beta+\gamma b\right]^3}{\left[-(n+2)(n-1)^2\beta^4-6(n-1)\beta^3+Z_1+2Z_2-Z_3\right]^2}Q,$$

where $Z_1 \equiv \left[\left(n^2+4n-1\right)\gamma b+3\left(n-2\right)\right]\beta^2$, $Z_2 \equiv 2\left[\gamma b(1-2n)+1\right]\beta$ and $Z_3 \equiv \gamma b(1-\gamma bn)$. The second order condition requires that $\gamma b > 1/2$ (see Table A2), then $2(\beta-1)\beta+\gamma b > 0$ for any $\beta \in [0,1]$, and as a result: $d^2W/d\lambda^2\big|_{\lambda=\hat{\lambda}_{AJ}} < 0$. Since $\hat{\lambda}_{AJ}$ is the unique stationary point of W, it follows that $\hat{\lambda}_{AJ}$ is a global maximum. This is the desired λ^o_{TS}.

Case KMZ: By inserting equilibrium values into the welfare function we get

$$W = \frac{1}{2}n\gamma(a-\bar{c})^2\frac{(2\Lambda+n)B\gamma b-\tau^2}{[(\Lambda+n)\gamma b-\tau]^2B}.$$

By differentiating W with respect to λ we obtain:

$$\frac{dW}{d\lambda} = -\frac{(n-1)(a-\bar{c})\gamma b\left\{\Lambda B\gamma b+\beta\left[2\lambda\left(B-n\right)+n-2-\beta(n+2)(n-1)\right]\right\}}{B\left[\left(\Lambda+n\right)b\gamma-\tau\right]^2}Q,$$

and by solving $dW/d\lambda = 0$ for λ we get a unique stationary point, given by $\hat{\lambda}_{KMZ}$. The second order derivative with respect to λ evaluated at $\lambda = \hat{\lambda}_{KMZ}$ yields

$$\left.\frac{d^2W}{d\lambda^2}\right|_{\lambda=\hat{\lambda}_{KMZ}} = \frac{\gamma b(n-1)^2(a-\bar{c})}{B\left[(\Lambda+n)\gamma b-\tau\right]^3}Z_{KMZ}Q,$$

where $Z_{KMZ} \equiv -\left[\beta n+(1-\beta)\right]n(\gamma b)^2+\left[4\beta(1-\beta)n+(1-\beta)^2-\beta^2n^2\right]\gamma b+\beta B\left[\beta(n+2)-2\right]$. The regulatory condition requires that $\gamma b > \tau/(\Lambda+n)$ (see Table A2), thus $d^2W/d\lambda^2\big|_{\lambda=\hat{\lambda}_{KMZ}} < 0$ whenever $Z_{KMZ} < 0$. Since $\hat{\lambda}_{KMZ}$ is the unique stationary point of W, it follows that $\hat{\lambda}_{KMZ}$ is a global maximum whenever $Z_{KMZ} < 0$. This is the desired λ^o_{TS}. It is straightforward to show that the regularity condition is stricter than the second order condition under the KMZ model specification for $n > 2$ (see Table A2). In addition, the regularity condition becomes stricter as λ and n increase. For $\lambda = 1$, the maximum value of the right-hand side of the regularity condition is $\sqrt{n}(n-1)/\left[4(n-\sqrt{n})\right]$, which for example equals 0.60 for $n = 2$ and 0.68 for $n = 3$. Numerical simulations show that assuming $\gamma b > 0.62$ guarantees that $Z_{KMZ} < 0$ holds for any n; thus, $Z_{KMZ} < 0$ is a mild condition: it is slightly stricter than the regularity condition in duopoly but softer for oligopoly of three or more firms.■

PROPOSITION A1 *A Research Joint Venture with no overlapping ownership (*$\lambda = 0$ *and* $\beta = 1$*) is socially optimal in AJ when* $\gamma b \geq n^2$*, in KMZ when* $\gamma b \geq n$*, and in CE* (provided that $W(\lambda)$ is single peaked) *when* $\alpha \geq \varepsilon n/[(n-1)\varepsilon^2 + (-1+n-2n^2)\varepsilon + n(n^2+1-n)]$*.*

Proof. When $W(\lambda)$ is single peaked, $\bar{\beta}$ is the minimum threshold above which allowing some positive λ is welfare enhancing (Proposition 4). Consequently, $\lambda^o_{\mathrm{TS}} = 0$ for any $\beta \in [0,1]$ if $\bar{\beta} \geq 1$. From Table A3 we have that $\bar{\beta}_{AJ} \geq 1$ if $\gamma b \geq n^2$ and $\bar{\beta}_{KMZ} \geq 1$ if $\gamma b \geq n$; in both cases $W(\lambda)$ is single peaked (see above). Also, from Table A3 we obtain $\bar{\beta}_{CE}$, and solving $\bar{\beta}_{CE} = 1$ for α, yields the threshold value in terms of n and ε: $\bar{\beta}_{CE} \geq 1$ if $\alpha \geq \varepsilon n/[(n-1)\varepsilon^2 + (-1+n-2n^2)\varepsilon + n(n^2+1-n)]$. Next we show that for $\lambda = 0$, $W'(\beta) > 0$ under AJ, KMZ and CE model specifications, and therefore it is socially optimal to set $\beta = 1$ in the three cases. We can write

$$\begin{aligned}\frac{\partial W}{\partial \beta} &= (f(Q^*)n - nc(Bx^*))\frac{\partial q^*}{\partial \beta} - nc'(Bx^*)(n-1)x^*q^* - nc'(Bx^*)B\frac{\partial x^*}{\partial \beta}q^* \qquad (57)\\ &\quad -n\Gamma'(x^*)\frac{\partial x^*}{\partial \beta}\\ &= \left[-\Lambda f'(Q^*)\frac{\partial q^*}{\partial \beta} - (1-\lambda)\beta(n-1)c'(Bx^*)\frac{\partial x^*}{\partial \beta} - c'(Bx^*)(n-1)x^*\right]Q^*.\end{aligned}$$

In AJ and for $\lambda = 0$, $\partial q^*/\partial \beta > 0$ and $\partial x^*/\partial \beta > 0$ (see Table A1), thus from (57) it is clear that $\partial W/\partial \beta > 0$. In KMZ and for $\lambda = 0$, $\partial q^*/\partial \beta = 0$ and $\partial x^*/\partial \beta < 0$. Higher R&D spillovers reduce R&D expenditures but also the unit cost of production of all firms. The latter dominates the former:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\lambda=0} = \frac{1}{2}\frac{n(a-\bar{c})^2\gamma(n-1)}{\left[b\gamma(n+1)-1\right]^2 B^2} > 0.$$

In CE and for $\lambda = 0$, $\partial q^*/\partial \beta = 0$ and $\partial x^*/\partial \beta < 0$. As in KMZ, welfare is increasing in β:

$$\left.\frac{\partial W}{\partial \beta}\right|_{\lambda=0} = \frac{n\left[\sigma\left(\frac{\alpha}{n}\right)^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon}{n}\right)\right]^{\frac{1}{\varepsilon-\alpha(1-\varepsilon)}}(n-1)}{B^2} > 0.■$$

Fig. A5 is a snapshot of the application and depicts optimal lambdas as a function of R&D spillovers in the first panel; welfare, consumer surplus and profit as a function of λ in the second panel, price and cost in the third panel, and q^* and x^* in the last panel (for $\beta = 0.5$ and $n = 6$). The figure illustrates that for an intermediate value of β, consumer surplus decreases with λ, and also does so welfare when λ is not too low (second panel), whereas for β sufficiently large, it is optimal in terms of consumer surplus and welfare to have $\lambda = 1$ (first panel).

Snapshot of the Application

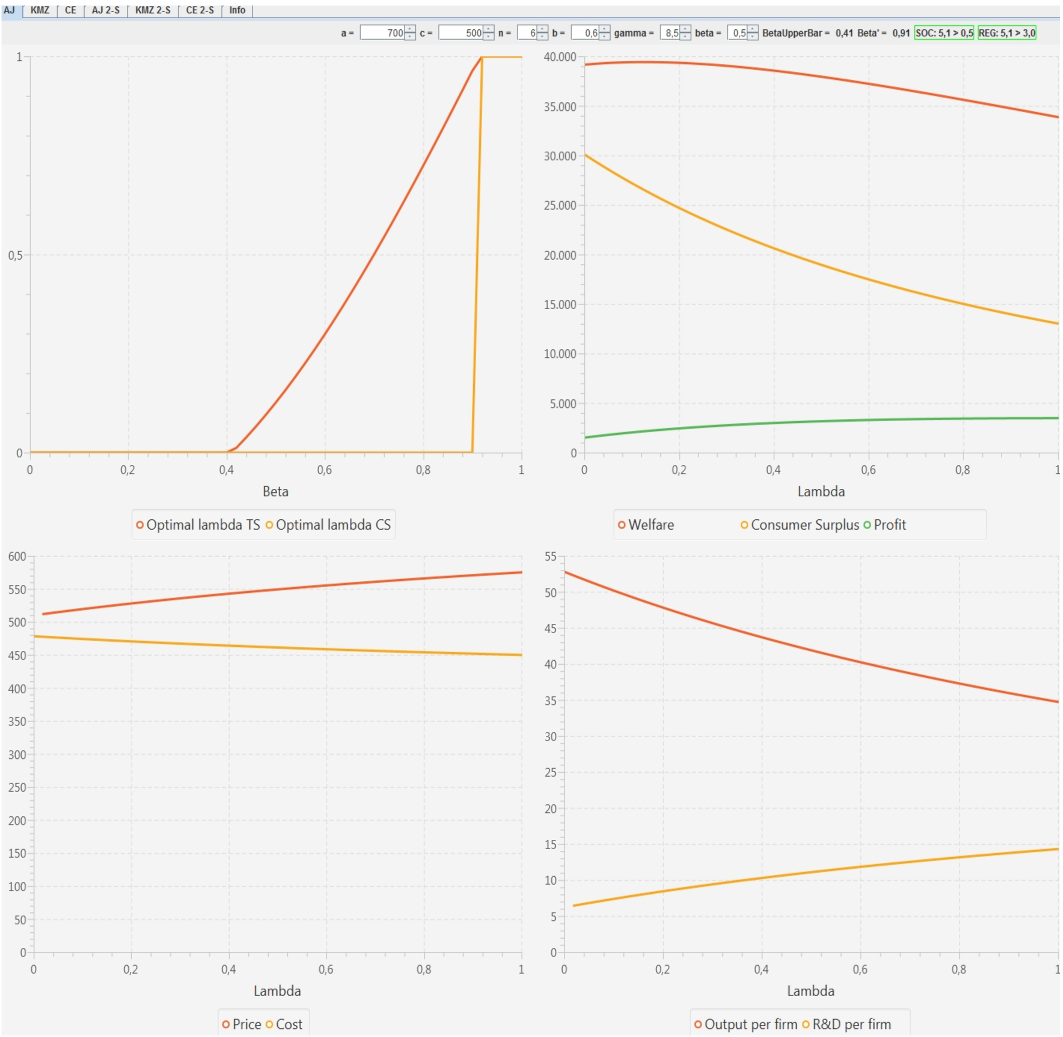

Fig. A5. AJ model. ($a = 700$, $\bar{c} = 500$, $\gamma = 8.5$, $\beta = 0.5$, $b = 0.6$)

Optimal degree of overlapping ownership (TS and CS standard)

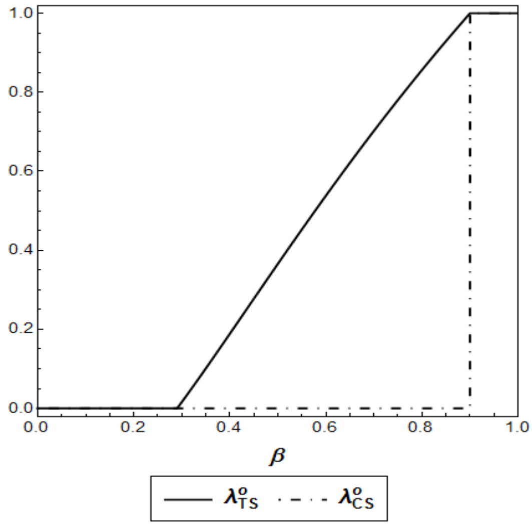

Fig. A6a. KMZ model.

($\gamma = 3$, $n = 6$, $b = 0.3$)

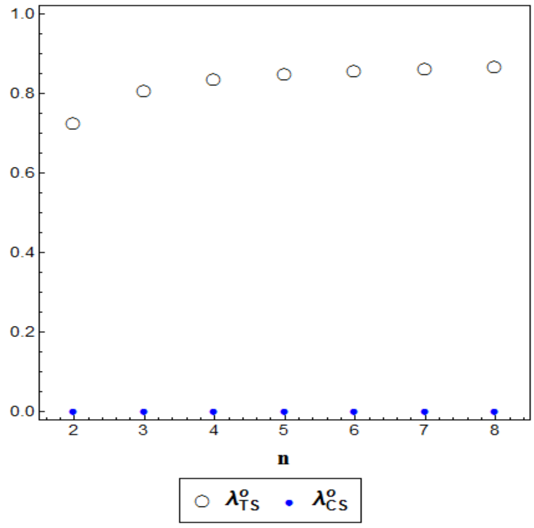

Fig. A6b. KMZ model.

($\gamma = 3$, $\beta = 0.8$, $b = 0.3$)

Table A5: Effect of Parameters on λ^o_{TS} and λ^o_{CS}

		λ^o_{TS}			λ^o_{CS}	
	AJ	KMZ	CE	AJ	KMZ	CE
Number of firms (n)	$+$	$+$	$+$	$\langle + \rangle$	0	$(+)$
Elasticity of demand (b^{-1}, ε^{-1})	$+$	$+$	$+$	$\langle + \rangle$	$\langle + \rangle$	$[+]$
Elasticity of innovation function (γ^{-1}, α)		$+$	$+$	$\langle + \rangle$	$\langle + \rangle$	$[+]$
Degree of spillover (β)	$+$	$+$	$+$	$(+)$	$(+)^*$	$[+]$

Key: $\langle + \rangle$, the parameter enlarges the region where $\lambda^o_{\text{CS}} = 1$; $(+)$, the effect is positive only if both β and n are sufficiently large (otherwise there is no effect); $(+)^*$, the effect is positive only if the parameter is sufficiently large and γb is sufficiently small (otherwise there is no effect); $[+]$, the effect is positive when n is sufficiently large (otherwise there is no effect).

A.2.2 Two-stage model

Next we present equilibrium values of output and R&D together with the expressions for sign$\{\partial q^*/\partial\lambda\}$ and sign$\{\partial x^*/\partial\lambda\}$ for each model specification. After that, we conduct a comparative statics analysis on $\bar{\beta}$, and on λ^o_{TS} and λ^o_{CS}. Finally, we compare the static and the two-stage model and briefly discuss the comparative statics on the other parameters of the model.

Equilibrium values and sign$\{\partial q^*/\partial\lambda\}$ **and** sign$\{\partial x^*/\partial\lambda\}$**.** We consider each case in turn.

Case AJ: FOCs (10; 14) yield

$$-b\Lambda q^* + a - bnq^* - \bar{c} + Bx^* = 0$$

$$\left[\tau + \frac{\Lambda}{n+\Lambda}(n-1)(1+\lambda-2\beta)\right] q^* - \gamma x^* = 0.$$

Solving the system for equilibrium values gives

$$q^* = \frac{\gamma(a-\bar{c})}{\tilde{\Delta}} \text{ and } x^* = \frac{\left[(n-1)(\frac{\Lambda}{n+\Lambda})(1+\lambda-2\beta)+\tau\right](a-\bar{c})}{\tilde{\Delta}}$$

where

$$\tilde{\Delta} \equiv \frac{\gamma b(\Lambda+n)^2 - B\left[(n-1)\Lambda(1+\lambda-2\beta)+(n+\Lambda)\tau\right]}{\Lambda+n}.$$

In this case, as in the simultaneous model, $H(\beta) = b\gamma$, then using (16) we obtain

$$\text{sign}\left\{\frac{\partial q^*}{\partial \lambda}\right\} = \text{sign}\left\{(B\beta - b\gamma)(n+\Lambda) + B\left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda\right]\right\}$$

and using (15) we get

$$\begin{aligned}\text{sign}\left\{\frac{\partial x^*}{\partial \lambda}\right\} &= \text{sign}\{\beta\left[\Lambda+n+(n-1)(\omega(\lambda)-\lambda)\right] \\ &+ \left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda\right] - 1 - (n-1)\omega(\lambda)\tilde{\beta}(\lambda)\},\end{aligned} \tag{58}$$

where we have used that

$$\left[\omega'(\lambda)(\tilde{\beta}(\lambda)-\beta)+\omega(\lambda)\tilde{\beta}'(\lambda)\right](\Lambda+n) = \frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n+\Lambda.$$

Case KMZ: The output and R&D values in equilibrium are given by (10; 14):

$$-b\Lambda q^* + a - bnq^* - \bar{c} + \left[\left(\frac{2}{\gamma}\right)Bx^*\right]^{1/2} = 0$$

$$\frac{1}{\gamma}\left[\left(\frac{2}{\gamma}\right)Bx^*\right]^{-1/2}\left[\tau + (n-1)\frac{\Lambda}{n+\Lambda}(1+\lambda-2\beta)\right] q^* - 1 = 0.$$

Solving the system for equilibrium values gives

$$q^* = \frac{\gamma(a-\bar{c})}{\gamma b(\Lambda+n) - \vartheta_{KMZ}} \text{ and } x^* = \frac{1}{2}\frac{(a-\bar{c})^2\vartheta_{KMZ}^2\gamma}{B\left[b\gamma(\Lambda+n)-\vartheta_{KMZ}\right]^2}$$

with $\vartheta_{KMZ} \equiv \tau + s(\lambda)(n-1) = (n-1)\frac{\Lambda}{n+\Lambda}(1+\lambda-2\beta)+\tau$, where $s(\lambda) \equiv \omega(\lambda)(\tilde{\beta}(\lambda)-\beta)$.

In this case, as in the simultaneous model, $H(\beta) = b\gamma B$, then from (16) we have

$$\text{sign}\left\{\frac{\partial q^*}{\partial \lambda}\right\} = \text{sign}\left\{(\beta - b\gamma)(n+\Lambda) + \left[\frac{1+\lambda-2\beta}{n+\Lambda}(n-1)n + \Lambda\right]\right\}$$

and $\text{sign}\{\frac{\partial x^*}{\partial \lambda}\}$ is again given by (58).

Case CE: The output and R&D values in equilibrium are obtained from (10; 14):

$$\sigma Q^{*-\varepsilon}\left(1 - \varepsilon\frac{\Lambda}{n}\right) - \kappa(Bx^*)^{-\alpha} = 0$$

$$\alpha(Bx^*)^{-\alpha-1}\left[\tau + (n-1)\omega(\lambda)(\tilde{\beta}(\lambda) - \beta)\right]\frac{Q^*}{n} = 1.$$

Solving the system for Q^* and x^*, after some manipulations, we get

$$Q^* = \frac{n}{\alpha\kappa\left[(n-1)s(\lambda)+\tau\right]}\left(\sigma\left\{\frac{\left[(n-1)s(\lambda)+\tau\right]\alpha}{n}\right\}^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon\Lambda}{n}\right)\right)^{(1+\alpha)/[\varepsilon-\alpha(1-\varepsilon)]}$$

and

$$x^* = \frac{1}{B}\left(\sigma\left\{\frac{\left[(n-1)s(\lambda)+\tau\right]\alpha}{n}\right\}^{\varepsilon}\kappa^{\varepsilon-1}\left(1-\frac{\varepsilon\Lambda}{n}\right)\right)^{1/[\varepsilon-\alpha(1-\varepsilon)]},$$

where $s(\lambda) \equiv \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)$ with

$$\omega(\lambda) = \frac{\Lambda\left[2n - \Lambda(1+\varepsilon)\right]}{n(n-\varepsilon\Lambda)} \quad \text{and} \quad \tilde{\beta}(\lambda) = \frac{n(1+\lambda) - \Lambda(1+\varepsilon)}{2n - \Lambda(1+\varepsilon)}.$$

Hence, we have

$$\text{sign}\left\{\frac{\partial q^*}{\partial \lambda}\right\} = \text{sign}\left\{\left[\beta + s'(\lambda)\right] - \frac{\alpha+1}{\alpha}\frac{\varepsilon}{n-\varepsilon\Lambda}\left[(n-1)s(\lambda)+\tau\right]\right\}.$$

And, one can obtain $\text{sign}\{\partial x^*/\partial\lambda\}$ by inserting values into (15) with $\delta = -(1+\varepsilon)$.

Threshold value $\bar{\beta}$

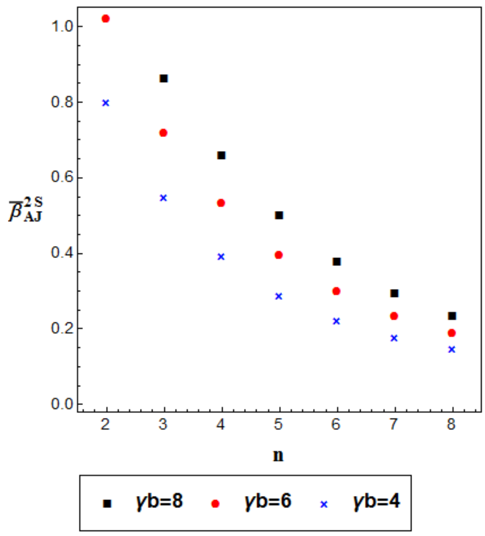

Fig. A7a. AJ model.

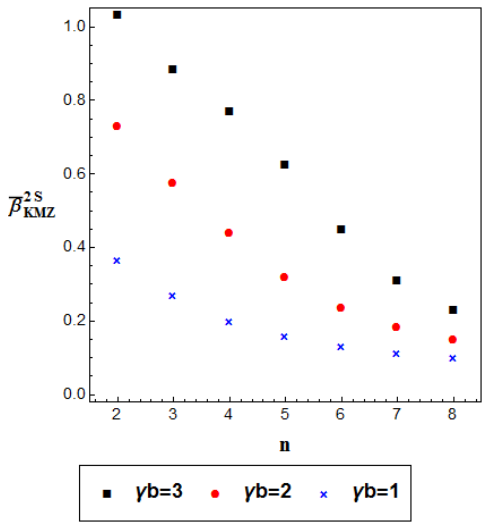

Fig. A7b. KMZ model.

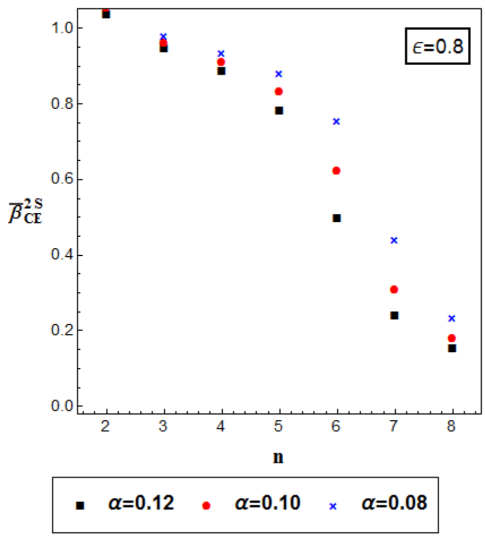

Fig. A8a. CE model.

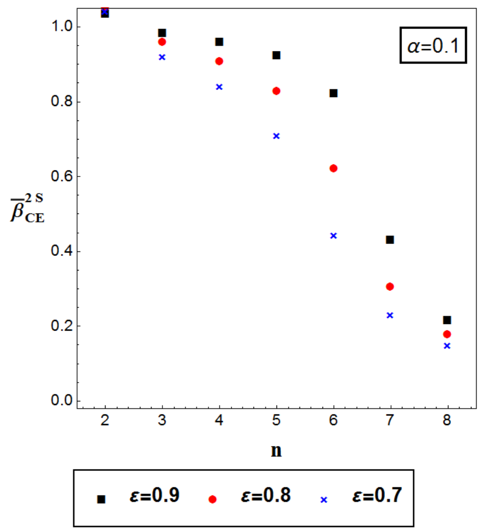

Fig. A8b. CE model.

Comparative statics on $\bar{\beta}$**.** Fig. A7a and A7b depict, respectively, the threshold $\bar{\beta}^{2S}$ under the AJ and KMZ model specifications. Fig. A7b reveals that in KMZ, $\bar{\beta}^{2S}$ tends to be above 1 if we consider the same values as in AJ. In particular, only if γb is low enough, we have that $\bar{\beta}^{2S} < 1$ (this result is in line with the simultaneous model). Also, we observe that under the AJ and KMZ model specifications, $\bar{\beta}^{2S}$ decreases with the number of firms and increases with γb. Figures A8a (respectively A8b) depict the threshold $\bar{\beta}^{2S}$ for the CE model for a given ε (α) and different values of n and α (ε). As in the simultaneous model, the threshold value decreases with n, the elasticity of the innovation function, α, and the elasticity of demand ε^{-1}.

Comparative statics on λ^o_{TS} and λ^o_{CS}. Fig. A9 is a snapshot of the application and plots welfare, consumer surplus, profit, price, cost, q^* and x^* as functions of λ (for $\beta = 0.65$ and $n = 6$). Note that profit at equilibrium can decrease with λ for λ sufficiently high. The reason is that in the two-stage model, there are incentives to overinvest so as to reduce the rival's output when β is not too high and this situation is more likely for larger values of λ since $\tilde{\beta}\ (\lambda)$ is increasing in λ. Recall that as $\lambda \to 1$ and $\beta < 1$, the symmetric equilibrium does not converge to the cartel outcome which yields profits which are higher than $\pi^*(\lambda)$ for any $\lambda < 1$. We have that $\pi^*(\lambda)$ converges to the monopoly profit as $\lambda \to 1$ when $\beta = 1$ (see the explanation just before the proof of Lemma 3). Fig. A10a, A10b and A10c show, respectively, optimal lambdas in AJ, KMZ and CE as functions of the number of firms. We see that under the three model specifications, λ^o_{TS} weakly increases with n, whereas λ^o_{CS} jumps with n only in AJ (and only for n sufficiently large).

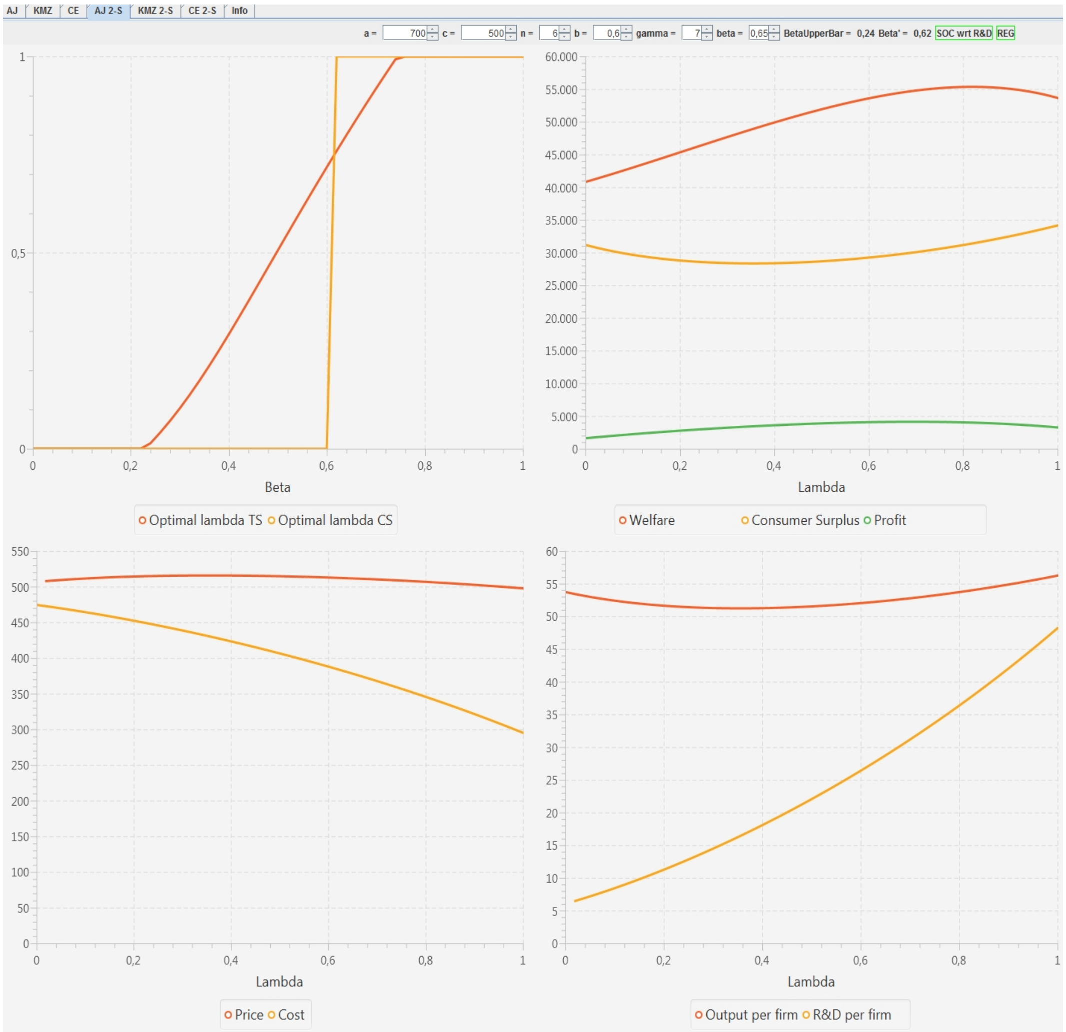

Fig. A9. AJ model. $(a = 700,\ \bar{c} = 500,\ \gamma = 7,\ n = 6,\ b = 0.6)$

Optimal degree of overlapping ownership (TS and CS standard)

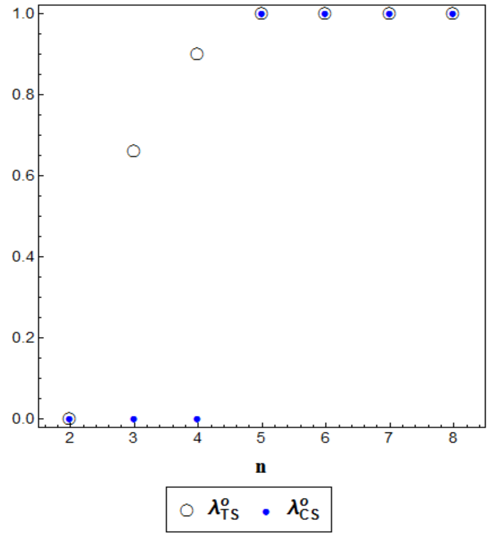

Fig. A10a. AJ model.
($a = 700$, $\bar{c} = 500$, $\gamma = 7$, $\beta = 0.8$, $b = 0.6$)

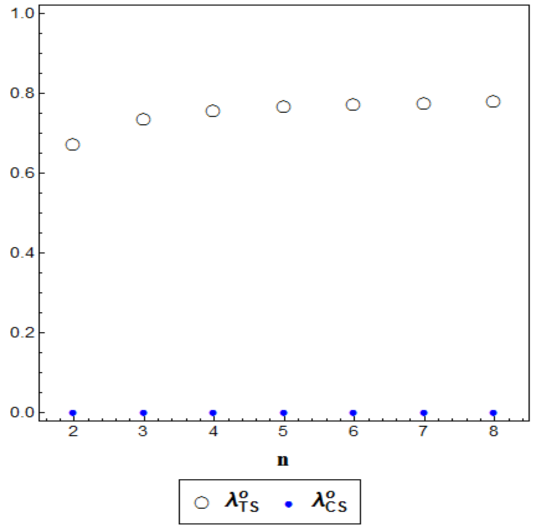

Fig. A10b. KMZ model.
($a = 700$, $\bar{c} = 500$, $\gamma = 5$, $\beta = 0.8$, $b = 0.3$)

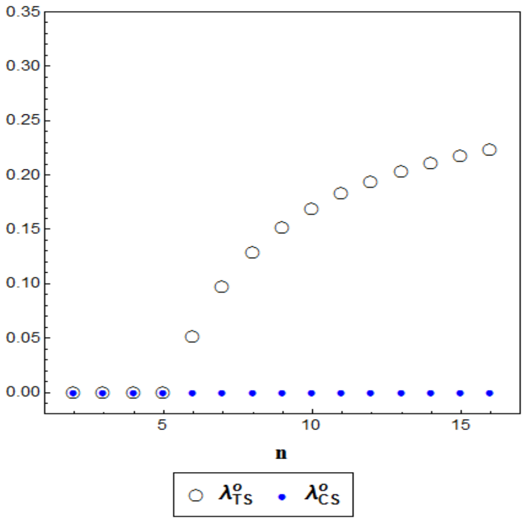

Fig. A10c. Constant elasticity model.
($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $\beta = 0.8$)

Comparison between the static and the two-stage model. In the constant elasticity model, as in the simultaneous case, we observe that if n is small then the equilibrium is in R_{I}, which implies that no overlapping ownership is socially optimal. Yet as β and n increase, λ^o_{TS} also increases.[6] Note that λ^o_{TS} in the two-stage game is above the static level in a large region of

[6]This result is consistent with the literature. For example, in a model with no overlapping ownership Spence

spillovers. For low values of β, the strategic effect is positive. Then, the two-stage model behaves differently than the static model in that welfare can *increase* with λ in R_{I} because it reduces R&D overinvestment by firms. This case is illustrated in Figure A11, where—for low β—λ^o_{TS} in the two-stage model is larger than in the static model. For intermediate values of spillovers, the strategic effect becomes negative (but remains close to zero); for higher spillover values, λ^o_{TS} increases with β more rapidly (i.e., convexly) when the strategic effect is strong.

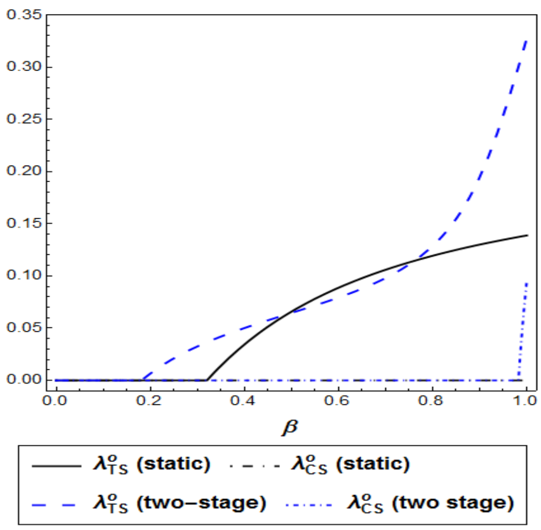

Fig. A11. Constant elasticity model.

($\alpha = 0.1$, $\varepsilon = 0.8$, $\sigma = \kappa = 1$, $n = 8$.)

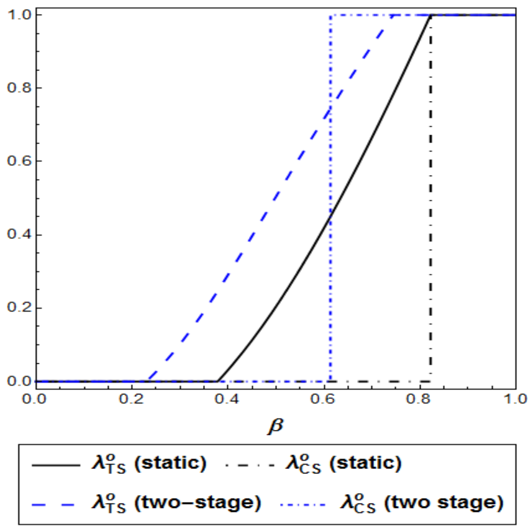

Fig A12. AJ model.

($\gamma = 7$, $n = 6$, $b = 0.6$.)

In the AJ model, we find that λ^o_{TS} and λ^o_{CS} are weakly larger in the two-stage case (see Figure A12). In contrast with the static model, the simulations indicate (for $\beta = 0.65$ and $n = 6$) that prices may be hump-shaped while cost decreases with λ; correspondingly, output per firm is U-shaped when R&D per firm increases. The welfare translation of the increase in λ displays U-shaped consumer surplus and increasing profit per firm, which results in an interior solution for welfare that features a large positive value of λ^o_{TS} (see Figure A9) with $\lambda^o_{\mathrm{CS}} = 1 > \lambda^o_{\mathrm{TS}} > 0$.

This becomes possible when the strategic effect is positive and strong enough. Then there is overinvestment in R&D during the first stage, which boosts output in the second stage. The strategic effect becomes positive for intermediate values of β when λ is sufficiently high. For an intermediate level of spillovers, total surplus is not maximized with full cooperation because that would entail too much production (reducing firms' profits).[7]

(1984) used numerical simulations to demonstrate that an increase in β reduces x^* and that, for a given β and $n \geq 2$, the cost reduction relative to the social optimum declines with n (see Spence 1984, Table I). It is socially good then to increase the degree of profit internalization.

[7] More precisely, since $\beta^{2S\prime}$ decreases with λ, it follows that—for a given β and a sufficiently high λ—we have $\beta > \beta^{2S\prime}$ and so the equilibrium is then in R_{III}, where CS increases with λ (CS is strictly convex in λ and so

Figure A13 shows optimal lambdas for KMZ as a function of β in the simultaneous and two-stage model. As in AJ, we can have $\lambda^o_{\text{CS}} > \lambda^o_{\text{TS}}$ for intermediate spillover values (because of the strategic effect).

Optimal degree of overlapping ownership (TS and CS standard)

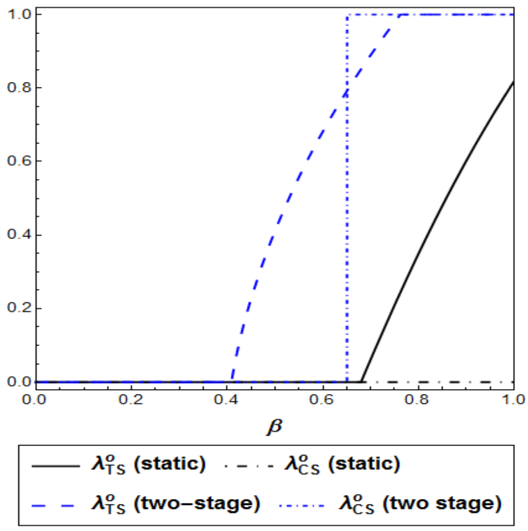

Fig. A13. KMZ model.

$(a = 700,\ \bar{c} = 500,\ \gamma = 5.5,\ n = 2,$

$b = 0.2)$

The pattern of results in our comparative statics analysis of the other parameters in AJ, KMZ, and CE is similar to that for the one-stage game (see Table A5). The only exceptions we have found are as follows. In AJ: although decreasing b enlarges the region where $\lambda^o_{\text{CS}} = 1$ is optimal (as in the static case), λ^o_{CS} can be lower than 1 (for a sufficiently low b) when spillovers are sufficiently high. In KMZ: although λ^o_{CS} is independent of n in the static case, in the two-stage game it can decrease with n when there are few firms in the market.

B Bertrand competition with differentiated products

B.1 Framework and equilibrium

In this Section we establish the framework and solve for the interior equilibrium of the model by deriving the FOCs.

$\lambda^*_{\text{CS}} = 1$ when $\text{CS}(1) > \text{CS}(0)$). In particular: for $\beta = 0.62$, the equilibrium is in R_{III} when $\lambda > 0.41$. Here the strategic effect is positive since $\tilde{\beta}(\lambda) > 0.62$ for $\lambda > 0.24$. Furthermore, if $\lambda > 0.69$ then the strategic effect is strong enough to *reverse* the sign of the effect of $\partial x^*/\partial\lambda$ on $W'(\lambda)$ (i.e., to make it negative); as a result, in a neighborhood of $\beta = 0.62$ there is a global maximum for $W(\lambda)$: even if the equilibrium is in R_{III} we have that $W'(\lambda) < 0$ for high values of λ, which implies $\lambda^o_{\text{TS}} \in (0, 1)$.

We consider an industry with n differentiated products, each produced by one firm. The demand for good i is given by $q_i = D_i(\mathbf{p})$ where $\mathbf{p}$ is the vector of prices. Goods are (strict) gross substitutes, $\partial D_i/\partial p_j > 0$, $j \neq i$. Assumptions A.2, A.3 and A.4 (with H as defined below) are maintained, we replace Assumption A.1 by the following one:

Assumption 1B. *For any product i, the function $D_i(\cdot)$ is smooth whenever positive, downward sloping, products are (strict) gross substitutes with $\partial D_i/\partial p_j > 0$, $j \neq i$, and the demand system $D(\cdot)$ is symmetric with negative definite Jacobian.*

Under Assumption 1B the demand system can be obtained from a representative consumer with quasilinear utility and can be inverted to obtain inverse demands (see Vives 1999, pp. 144-148). Furthermore, it follows that the demand for a variety when all firms set the same price (the Chamberlinian DD function) is downward sloping since the own-price effect dominates the cross-price effects:

$$v \equiv \frac{\partial D_i}{\partial p_i} + (n-1)\frac{\partial D_j}{\partial p_i} < 0,\ j \neq i.$$

It follows that $v_\lambda \equiv \partial D_i/\partial p_i + \lambda(n-1)\partial D_k/\partial p_i < 0$. The innovation function is defined as in Cournot. The firm i's profit now writes as

$$\pi_i = \left(p_i - c\left(x_i + \beta\sum_{j\neq i} x_j\right)\right) D_i(\mathbf{p}) - \Gamma(x_i)$$

and the objective function for the manager of firm i is again: $\phi_i = \pi_i + \lambda\sum_{k\neq i}\pi_k$, thus

$$\phi_i = \left(p_i - c\left(x_i + \beta\sum_{j\neq i} x_j\right)\right) D_i(\mathbf{p}) - \Gamma(x_i) + \lambda\sum_{k\neq i}\left[\left(p_k - c\left(x_k + \beta\sum_{j\neq k} x_j\right)\right) D_k(\mathbf{p}) - \Gamma(x_k)\right].$$

B.2 Simultaneous model

The FOCs for an interior symmetric equilibrium are

$$\frac{\partial \phi_i}{\partial p_i} = D_i(\mathbf{p}) + (p_i - c_i)\frac{\partial D_i(\mathbf{p})}{\partial p_i} + \lambda\sum_{k\neq i}(p_k - c_k)\frac{\partial D_k(\mathbf{p})}{\partial p_i} = 0, \tag{59}$$

$$\frac{\partial \phi_i}{\partial x_i} = -c'(\cdot)D_i(\mathbf{p}) - \Gamma'(x_i) - \lambda\sum_{k\neq i} c'(\cdot)\beta D_k(\mathbf{p}) = 0. \tag{60}$$

The symmetric equilibrium is the pair (p^*, x^*), with $q^* = D_i(\mathbf{p}^*)$ where $\mathbf{p}^* = (p^*, ..., p^*)$ for all i, that solves the system (59)-(60). The FOC for price in the symmetric equilibrium is

$$q^* + (p^* - c(Bx^*))\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)(p^* - c(Bx^*))\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} = 0.$$

Note that $v < 0$ ensures that $p^* - c(Bx^*)$ is strictly positive for all λ; the above condition can be rewritten as

$$q^* + \frac{(p^* - c(Bx^*))}{p^*}\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i}\frac{p^*q^*}{D_i(\mathbf{p}^*)} + \lambda(n-1)\frac{(p^* - c(Bx^*))}{p^*}\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\frac{p^*q^*}{D_k(\mathbf{p}^*)} = 0.$$

Using the notation: $\eta_i = -\left(\partial D_i(\mathbf{p}^*)/\partial p_i\right)\left(p^*/D_i(\mathbf{p}^*)\right)$ and $\eta_{ik} = \left(\partial D_k(\mathbf{p}^*)/\partial p_i\right)\left(p^*/D_k(\mathbf{p}^*)\right)$, $k \neq i$, we can write

$$1 - \frac{p^* - c(Bx^*)}{p^*}\eta_i + \lambda(n-1)\frac{p^* - c(Bx^*)}{p^*}\eta_{ik} = 0.$$

From the above condition and from (60), a symmetric (interior) equilibrium will satisfy the following two conditions:

$$\frac{p^* - c(Bx^*)}{p^*} = \frac{1}{\eta_i - \lambda(n-1)\eta_{ik}}; \tag{61}$$

$$-c'(Bx^*)q^*\tau = \Gamma'(x^*). \tag{62}$$

Note that the latter condition is also obtained in Cournot oligopoly.

Finally, we assume the following parallel regularity conditions to the Cournot case:

$$\Delta_p \equiv \partial_{p_i p_i}\phi_i + (n-1)\partial_{p_i p_j}\phi_i < 0 \tag{63}$$

and

$$\Delta \equiv \Delta_p \Delta_x - \left[\partial_{x_i p_i}\phi_i + (n-1)\partial_{p_j x_i}\phi_i\right]\left[\partial_{x_i p_i}\phi_i + (n-1)\partial_{x_j p_i}\phi_i\right] > 0, \tag{64}$$

where

$$\Delta_x \equiv \partial_{x_i x_i}\phi_i + (n-1)\partial_{x_i x_j}\phi_i.$$

Since $\partial_{x_i x_i}\phi_i = -c''(Bx^*)\left[1 + \lambda(n-1)\beta^2\right]q^* - \Gamma''(x^*)$ and $\partial_{x_i x_j}\phi_i = -c''(Bx^*)\left[\tau + \lambda\left(1-\beta\right)\right]\beta q^*$, it follows that

$$\Delta_x = -c''(Bx^*)q^*\tau B - \Gamma''(x^*) < 0 \tag{65}$$

under Assumptions A.2 and A.3. Together conditions (63) and (64) imply that the FOCs (61) and (62) both have a unique symmetric solution if they hold globally, and we assume that a symmetric regular equilibrium exists.

B.2.1 Comparative statics with respect to λ

In this Section we show that, as in the Cournot oligopoly model, if $\partial x^*/\partial\lambda \leq 0$, then $\partial p^*/\partial\lambda > 0$ (Lemma B1). Secondly, we derive the signs: $\operatorname{sign}\{\partial x^*/\partial\lambda\}$ and $\operatorname{sign}\{\partial p^*/\partial\lambda\}$ (Lemma B2).

Finally, we discuss conditions that identify the three regions in Bertrand competition with product differentiation.

As in the Cournot oligopoly model, we can establish

LEMMA B1 *In the symmetric equilibrium,* $\frac{\partial p^*}{\partial \lambda} > 0$ *if* $\frac{\partial x^*}{\partial \lambda} \leq 0$.

Proof. By totally differentiating the FOC $\partial \phi_i / \partial p_i = 0$ with respect to λ we obtain:

$$\partial_{p_i p_i} \phi_i \frac{\partial p^*}{\partial \lambda} + (n-1)\partial_{p_i p_j} \phi_i \frac{\partial p^*}{\partial \lambda} + \partial_{x_i p_i} \phi_i \frac{\partial x^*}{\partial \lambda} + (n-1)\partial_{x_j p_i} \phi_i \frac{\partial x^*}{\partial \lambda} + \partial_{\lambda p_i} \phi_i = 0.$$

Therefore,

$$\frac{\partial p^*}{\partial \lambda} = -\frac{1}{\partial_{p_i p_i} \phi_i + (n-1)\partial_{p_i p_j} \phi_i} \left\{ \partial_{\lambda p_i} \phi_i + \left[\partial_{x_i p_i} \phi_i + (n-1)\partial_{x_j p_i} \phi_i \right] \frac{\partial x^*}{\partial \lambda} \right\}.$$

Using the stability condition $\Delta_p < 0$, it follows that

$$\operatorname{sign} \left\{ \frac{\partial p^*}{\partial \lambda} \right\} = \operatorname{sign} \left\{ \partial_{\lambda p_i} \phi_i + \left[\partial_{x_i p_i} \phi_i + (n-1)\partial_{x_j p_i} \phi_i \right] \frac{\partial x^*}{\partial \lambda} \right\}. \tag{66}$$

Since $\partial_{\lambda p_i} \phi_i = (n-1)(p^* - c(Bx^*))\partial D_k(\mathbf{p}^*)/\partial p_i > 0$, we have that

$$\frac{\partial x^*}{\partial \lambda} \leq 0 \Rightarrow \frac{\partial p^*}{\partial \lambda} > 0 \text{ when } \vartheta \equiv \partial_{x_i p_i} \phi_i + (n-1)\partial_{x_j p_i} \phi_i < 0.$$

Note that

$$\begin{aligned} \partial_{x_i p_i} \phi_i &= -c'(Bx^*) \frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} - \lambda(n-1)c'(Bx^*)\beta \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \\ &= -\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)\beta \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \right] c'(Bx^*). \end{aligned}$$

The expression $\partial_{x_j p_i} \phi_i$ can be obtained from (59):

$$\begin{aligned} \partial_{x_j p_i} \phi_i &= -c'(Bx^*)\beta \frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} - \lambda c'(Bx^*) \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} - \lambda(n-2)c'(Bx^*)\beta \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \\ &= -\left[\beta \frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda (B - \beta) \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \right] c'(Bx^*). \end{aligned}$$

Using the above expression we can write

$$\begin{aligned}
\vartheta &= -\left\{\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)\beta\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} + (n-1)\left[\beta\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda\left(B-\beta\right)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right]\right\} c'(Bx^*) \\
&= -\left\{B\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + (n-1)\left[\lambda\beta + \lambda\left(B-\beta\right)\right]\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right\} c'(Bx^*) \\
&= -\left[B\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)B\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right] c'(Bx^*) \\
&= -B\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right] c'(Bx^*).
\end{aligned}$$

Assumptions A.2 and $v < 0$ imply that $\vartheta < 0$.■

By totally differentiating the FOCs with respect to λ and solving for $\partial p^*/\partial\lambda$ and $\partial x^*/\partial\lambda$ we obtain:

$$\frac{\partial p^*}{\partial\lambda} = \frac{1}{\Delta}\left\{\partial_{\lambda x_i}\phi_i\left[\partial_{x_i p_i}\phi_i + (n-1)\partial_{x_j p_i}\phi_i\right] - \partial_{\lambda p_i}\phi_i\Delta_x\right\} \tag{67}$$

and

$$\frac{\partial x^*}{\partial\lambda} = \frac{1}{\Delta}\left\{\partial_{\lambda p_i}\phi_i\left[\partial_{x_i p_i}\phi_i + (n-1)\partial_{p_j x_i}\phi_i\right] - \partial_{\lambda x_i}\phi_i\Delta_p\right\}. \tag{68}$$

To obtain sign $\{\partial x^*/\partial\lambda\}$ and sign $\{\partial p^*/\partial\lambda\}$ we next derive in turn each of the expressions contained in equations (67) and (68). After some manipulations we can establish:

$$\partial_{x_i p_i}\phi_i + (n-1)\partial_{x_j p_i}\phi_i = -Bv_\lambda c'(Bx^*),$$

$$\partial_{x_i p_i}\phi_i + (n-1)\partial_{p_j x_i}\phi_i = -\tau v c'(Bx^*).$$

We also have that

$$\partial_{\lambda x_i}\phi_i = -(n-1)c'(Bx^*)\beta q^* \geq 0,$$

$$\partial_{\lambda p_i}\phi_i = (n-1)(p^* - c(Bx^*))\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} > 0.$$

Finally, we need the expressions for Δ_p (the expression for Δ_x is given by (65)). Recall that $\Delta_p \equiv \partial_{p_i p_i}\phi_i + (n-1)\partial_{p_i p_j}\phi_i$. By differentiating and evaluating in the symmetric equilibrium, we obtain

$$\partial_{p_i p_i}\phi_i = 2\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + (p^* - c(Bx^*))\left[\frac{\partial^2 D_i(\mathbf{p}^*)}{\partial p_i^2} + \lambda(n-1)\frac{\partial^2 D_k(\mathbf{p}^*)}{\partial p_i^2}\right]$$

and, using that in the symmetric equilibrium $\partial D_i/\partial p_j = \partial D_j/\partial p_i$ and $\partial^2 D_i/\partial p_j\partial p_i = \partial^2 D_j/\partial p_j\partial p_i$,

$$\partial_{p_i p_j}\phi_i = (1+\lambda)\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j} + (p^* - c(Bx^*))\left[(1+\lambda)\frac{\partial^2 D_i(\mathbf{p}^*)}{\partial p_j\partial p_i} + \lambda(n-2)\frac{\partial^2 D_k(\mathbf{p}^*)}{\partial p_j\partial p_i}\right]. \tag{69}$$

Thus,

$$\Delta_p = v + v_\lambda - \frac{q^*}{v_\lambda}(n-1)\left\{\frac{1}{n-1}\frac{\partial^2 D_i(\mathbf{p}^*)}{\partial p_i^2} + \lambda\frac{\partial^2 D_k(\mathbf{p}^*)}{\partial p_i^2}\right. \\ \left. + \left[(1+\lambda)\frac{\partial^2 D_i(\mathbf{p}^*)}{\partial p_j \partial p_i} + \lambda(n-2)\frac{\partial^2 D_k(\mathbf{p}^*)}{\partial p_j \partial p_i}\right]\right\}. \tag{70}$$

Therefore,

$$\Delta = -\Delta_p \left(c''(Bx^*)q^*\tau B + \Gamma''(x^*)\right) - \tau B v v_\lambda \left(c'(Bx^*)\right)^2.$$

Under regularity condition $\Delta > 0$, then:

$$\text{sign}\left\{\frac{\partial x^*}{\partial \lambda}\right\} = \text{sign}\left\{\tau\left(p^* - c(Bx^*)\right)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}v - \beta q^* \Delta_p\right\} \tag{71}$$

and

$$\text{sign}\left\{\frac{\partial p^*}{\partial \lambda}\right\} = \text{sign}\left\{-(n-1)c'(Bx^*)\beta q^*\left[-B v_\lambda c'(Bx^*)\right] - (n-1)(p^* - c(Bx^*))\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\Delta_x\right\},$$

thus

$$\text{sign}\left\{\frac{\partial p^*}{\partial \lambda}\right\} = \text{sign}\left\{-B\beta q^* v_\lambda c'(Bx^*) + (p^* - c(Bx^*))\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\frac{\Delta_x}{c'(Bx^*)}\right\}. \tag{72}$$

Clearly, from (71) and (72), and in line with the Cournot oligopoly model: for $\beta = 0$, $\partial x^*/\partial\lambda < 0$ and $\partial p^*/\partial\lambda > 0$. Let $P'(c)$ be the cost pass-through coefficient $P'(c) \equiv dp^*/dc$; for $\beta > 0$ we can establish the analogous to Lemmata 1 and 2:

LEMMA B2 *In the symmetric equilibrium*

$$\text{sign}\left\{\frac{\partial x^*}{\partial \lambda}\right\} = \text{sign}\left\{\beta - P'(c)\frac{|v|}{v_\lambda^2}\tau\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right\}, \tag{73}$$

where $P'(c) = v_\lambda/\Delta_p > 0$, *and*

$$\text{sign}\left\{\frac{\partial p^*}{\partial \lambda}\right\} = \text{sign}\left\{H - \beta B\right\}, \tag{74}$$

where

$$H = \frac{\partial D_k(\mathbf{p}^*)/\partial p_i}{\left(v_\lambda c'(Bx^*)\right)^2}\left[-\frac{c''(Bx^*)B\Gamma'(x^*)}{c'(Bx^*)} + \Gamma''(x^*)\right]. \tag{75}$$

Proof. Inserting the FOC with respect to the price, $p^* - c(Bx^*) = -q^*/v_\lambda$, into (71) yields

$$\text{sign}\left\{\partial x^*/\partial\lambda\right\} = \text{sign}\left\{-\tau\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\left(\frac{v}{v_\lambda}\right) - \beta\Delta_p\right\}.$$

By computing the total derivative of $\partial\phi_i/\partial p_i = 0$ with respect to the cost c, we obtain $P'(c) =$

v_λ/Δ_p, and therefore (73). Using again the FOC: $p^* - c(Bx^*) = -q^*/v_\lambda$, and equation (65), we get

$$\operatorname{sign}\left\{\frac{\partial p^*}{\partial \lambda}\right\} = \operatorname{sign}\left\{-B\beta v_\lambda c'(Bx^*) - \frac{1}{v_\lambda c'(Bx^*)}\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\left[-c''(Bx^*)q^*(\tau B) - \Gamma''(x^*)\right]\right\}.$$

Noting that the FOC with respect to R&D investment can be re-written as $q^* = \Gamma'(x^*)/(-c'(Bx^*)\tau)$, and using that $v_\lambda c'(Bx^*) > 0$, we have

$$\operatorname{sign}\left\{\frac{\partial p^*}{\partial \lambda}\right\} = \operatorname{sign}\left\{-B + \frac{1}{\beta\left(v_\lambda c'(Bx^*)\right)^2}\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\left[-\frac{c''(Bx^*)B\Gamma'(x^*)}{c'(Bx^*)} + \Gamma''(x^*)\right]\right\}.$$

As in the Cournot oligopoly model we define the function H for Bertrand competition with differentiated products as shown in equation (75). Thus, in the symmetric equilibrium: $\operatorname{sign}\{\partial p^*/\partial\lambda\} = \operatorname{sign}\{H - \beta B\}$.$\blacksquare$

In Cournot we showed that $\operatorname{sign}\{\partial q^*/\partial\lambda\} = \operatorname{sign}\{\beta B - H\}$. The reverse of the terms inside the curly brackets is explained by the different type of competition (price/output competition) in the two models. Assuming that $\Gamma'' > 0$, we can rewrite H as follows:

$$H = \frac{\partial D_k(\mathbf{p}^*)/\partial p_i}{\left(v_\lambda c'(Bx^*)\right)^2}\Gamma''(x^*)\left[-\frac{c''(Bx^*)Bx^*}{c'(Bx^*)}\frac{\Gamma'(x^*)}{\Gamma''(x^*)x^*} + 1\right]. \tag{76}$$

By defining, as we did in the Cournot model, $\chi(Bx^*) \equiv -c''(Bx^*)Bx^*/c'(Bx^*) \geq 0$, $y(x^*) \equiv \Gamma''(x^*)x^*/\Gamma'(x^*) \geq 0$,

$$\xi(q^*, x^*) \equiv \frac{\left(v_\lambda c'(Bx^*)\right)^2}{\frac{\partial D_k(p^*)}{\partial p_i}\Gamma''(x^*)} > 0,$$

and by replacing these terms into (76) we get

$$H = \frac{1}{\xi(q^*, x^*)}\left(1 + \frac{\chi(Bx^*)}{y(x^*)}\right).$$

Note that the only difference with respect to the Cournot model is that here the expression for the relative effectiveness of R&D (ξ) takes into account the fact that products are now differentiated. In Cournot: $\xi = -(c'(Bx^*))^2/(f'(Q^*)\Gamma''(x^*))$; in Bertrand with differentiated products, however, the term $(f')^{-1}$ is replaced by ${v_\lambda}^2\left(\partial D_k(\mathbf{p}^*)/\partial p_i\right)^{-1}$.

We can proceed as in the Cournot model and define the corresponding three regions: R_{I}, where $\partial p^*/\partial\lambda > 0$ and $\partial x^*/\partial\lambda \leq 0$; R_{II} where $\partial p^*/\partial\lambda > 0$ and $\partial x^*/\partial\lambda > 0$; R_{III} where $\partial p^*/\partial\lambda < 0$ and $\partial x^*/\partial\lambda > 0$.

Regarding R_{I}, because of gross substitutes ($\partial D_k(\mathbf{p}^*)/\partial p_i > 0$), we can have $\partial x^*/\partial\lambda < 0$ for all β (73). This is the case when $-\Delta_p < \Lambda\left(\partial D_k(\mathbf{p}^*)/\partial p_i\right)v/v_\lambda$. Regarding the spillover threshold between R_{II} and R_{III}, note that here, as in Cournot, Assumption A.4 implies that the equation $H - \beta B = 0$ has a unique positive solution, which again we may denote by β'. It follows that for $\beta > \beta'$, $\partial p^*/\partial\lambda < 0$. Furthermore, R_{III} exists (because the threshold β' is strictly lower than 1) when $n > H(1)$.

B.2.2 Welfare analysis

Welfare (with quasilinear utility) at a symmetric equilibrium is given by

$$W = U(\mathbf{q}^*) - c(Bx^*)nq^* - n\Gamma(x^*),$$

where $\mathbf{q}^*$ is the equilibrium output vector and U is the utility of a representative consumer, assumed to be smooth and strictly concave (i.e., with a negative definite Hessian). By differentiating with respect to λ:

$$W'(\lambda) = \left(\sum_i \frac{\partial U(\mathbf{q}^*)}{\partial q_i} - nc(Bx^*)\right)\frac{\partial q^*}{\partial\lambda} - \left(nc'(Bx^*)Bq^* + n\Gamma'(x^*)\right)\frac{\partial x^*}{\partial\lambda}.$$

From the maximization problem of the consumer: $p_i = \partial U(\mathbf{q}^*)/\partial q_i$, so

$$W'(\lambda) = (p^* - c(Bx^*))n\frac{\partial q^*}{\partial\lambda} - \left(nc'(Bx^*)Bq^* + n\Gamma'(x^*)\right)\frac{\partial x^*}{\partial\lambda}.$$

From the FOC with respect to price: $p^* - c(Bx^*) = -q^*/v_\lambda$, and from the FOC with respect to R&D investment: $\Gamma'(x^*) = -c'(Bx^*)q^*\tau$, thus

$$\begin{aligned} W'(\lambda) &= -\frac{q^*}{v_\lambda}n\frac{\partial q^*}{\partial\lambda} - \left(nc'(Bx^*)Bq^* - nc'(Bx^*)q^*\tau\right)\frac{\partial x^*}{\partial\lambda} \\ &= -\frac{q^*}{v_\lambda}n\frac{\partial q^*}{\partial\lambda} - nc'(Bx^*)q^*(B-\tau)\frac{\partial x^*}{\partial\lambda}. \end{aligned}$$

From the demand definition, $q^* = D_i(p^*(\lambda))$ we have that $\partial q^*/\partial\lambda = v\,(\partial p^*/\partial\lambda)$. Using that $B - \tau = (1-\lambda)\beta(n-1)$, we finally may write

$$W'(\lambda) = -\left[\frac{v}{v_\lambda}\frac{\partial p^*}{\partial\lambda} + (1-\lambda)\beta(n-1)c'(Bx^*)\frac{\partial x^*}{\partial\lambda}\right]nq^*. \tag{77}$$

Thus,

- in R_{I}, where $\partial x^*/\partial\lambda \leq 0$ and $\partial p^*/\partial\lambda > 0$ (so $\partial q^*/\partial\lambda < 0$): $W'(\lambda) < 0$.
- in R_{II}, where $\partial x^*/\partial\lambda > 0$ and $\partial p^*/\partial\lambda > 0$ (so $\partial q^*/\partial\lambda < 0$): $W'(\lambda) \lesseqgtr 0$.
- in R_{III}, where $\partial x^*/\partial\lambda > 0$ and $\partial p^*/\partial\lambda < 0$ (so $\partial q^*/\partial\lambda > 0$): $W'(\lambda) > 0$.

From (67), it follows that

$$\frac{\partial p^*}{\partial \lambda} = \frac{(n-1)q^*}{\Delta}\left\{\left(c'(Bx^*)\right)^2 \beta B v_\lambda - \frac{\partial D_k(\mathbf{p}^*)/\partial p_i}{v_\lambda}\left[c''(Bx^*)q^* B\tau + \Gamma''(x^*)\right]\right\}. \tag{78}$$

Similarly, from (68), after some manipulations, we obtain

$$\frac{\partial x^*}{\partial \lambda} = \frac{(n-1)q^*(-c'(Bx^*))}{\Delta}\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\left[\beta\left(\frac{\Delta_p}{-\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}}\right) - \frac{v}{v_\lambda}\tau\right]. \tag{79}$$

By inserting (78) and (79) into (77) we obtain

$$W'(\lambda) = \frac{(n-1)n\; {q^*}^2}{\Delta}\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\mathcal{F} \tag{80}$$

where

$$\begin{aligned}\mathcal{F} \quad \equiv \quad & \frac{\left(c'(Bx^*)\right)^2 \beta B(-v)}{\partial D_k(\mathbf{p}^*)/\partial p_i} + \frac{v}{{v_\lambda}^2}\left[c''(Bx^*)q^* B\tau + \Gamma''(x^*)\right] \\ & + \left(c'(Bx^*)\right)^2 (1-\lambda)\beta(n-1)\left[\frac{\beta\left(-\Delta_p\right)}{\partial D_k(\mathbf{p}^*)/\partial p_i} - \frac{v}{v_\lambda}\tau\right].\end{aligned} \tag{81}$$

Remark B1. Consider the case of independent products, $\partial D_k(\mathbf{p}^*)/\partial p_i = 0$. If the local monopoly problem is well-defined we have: (i) if $\beta > 0$, then $\lambda^o_{\mathrm{TS}} = \lambda^o_{\mathrm{CS}} = 1$, whereas (ii) if $\beta = 0$, then λ has no impact on total surplus or consumer surplus.

Proof. It follows immediately from equation (71) that $\operatorname{sign}\{\partial x^*/\partial\lambda\} > 0$ for $\beta > 0$ and $\partial x^*/\partial\lambda = 0$ for $\beta = 0$. Similarly, from equation (72): $\operatorname{sign}\{\partial p^*/\partial\lambda\} < 0$ (or equivalently $\operatorname{sign}\{\partial q^*/\partial\lambda\} > 0$) for $\beta > 0$, while $\partial p^*/\partial\lambda = \partial q^*/\partial\lambda = 0$ for $\beta = 0$. Using (77), $W'(\lambda) > 0$ for all λ if $\beta > 0$, thus $\lambda^o_{\mathrm{TS}} = 1$. Since $\operatorname{sign}\{CS'(\lambda)\} = \operatorname{sign}\{\partial q^*/\partial\lambda\}$, we also have that $\lambda^o_{\mathrm{CS}} = 1$. If $\beta = 0$, clearly from (77), $W'(\lambda) = 0$; note that for $\partial D_k(\mathbf{p}^*)/\partial p_i = \beta = 0$, FOCs do not depend on λ.■

B.3 Two-stage model

We first derive the FOCs and the expression for $\tilde{\beta}(\lambda)$ for the Bertrand case. We then discuss the strategic effect and welfare in Bertrand with two stages.

Interior equilibrium and threshold $\tilde{\beta}(\lambda)$. Let

$$\varphi \equiv -\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i}\partial_{p_i p_i}\phi_i + \lambda\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\left[(n-1)\partial_{p_i p_j}\phi_i - (n-2)\partial_{p_i p_i}\phi_i\right].$$

Then, using (36) we can write:

$$\frac{\partial}{\partial x_i} p_j^*(\mathbf{x}) = \frac{-c'(Bx)}{\Omega} (-\varphi) \left(\tilde{\beta}(\lambda) - \beta \right), \tag{82}$$

where

$$\Omega \equiv \left(\partial_{p_i p_i} \phi_i - \partial_{p_i p_j} \phi_i \right) \left[\partial_{p_i p_i} \phi_i + (n-1) \, \partial_{p_i p_j} \phi_i \right] \tag{83}$$

and

$$\tilde{\beta}(\lambda) = \frac{1}{(-\varphi)} \left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_j} \phi_i - \lambda \frac{\partial D_j(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_i} \phi_i \right]. \tag{84}$$

The denominator of $\tilde{\beta}(\lambda)$ is positive since $\varphi < 0$:

$$\begin{aligned} \varphi & \equiv -\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_i} \phi_i + \lambda \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \left[(n-1) \partial_{p_i p_j} \phi_i - (n-2) \partial_{p_i p_i} \phi_i \right] \qquad (85) \\ & = -\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_i} \phi_i + \lambda \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \left[\partial_{p_i p_i} \phi_i + (n-1) \partial_{p_i p_j} \phi_i \right] - \lambda \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} (n-1) \partial_{p_i p_i} \phi_i \\ & = \lambda \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \Delta_p - \partial_{p_i p_i} \phi_i \left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda (n-1) \frac{\partial D_k(\mathbf{p}^*)}{\partial p_i} \right] < 0. \end{aligned}$$

Therefore, if

$$\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_j} \phi_i - \lambda \frac{\partial D_j(\mathbf{p}^*)}{\partial p_i} \partial_{p_i p_i} \phi_i < 0 \tag{86}$$

then $\tilde{\beta}(\lambda) < 0$. Condition (86) is satisfied in the case of linear and constant elasticity demand with differentiated products (see analysis below).

Finally, note that in Bertrand at the symmetric equilibrium FOCs boil down to

$$q^* + (p^* - c(Bx^*)) v_\lambda = 0 \tag{87}$$

and

$$-c'(Bx^*) \tau q^* - \Gamma'(x^*) + (n-1) \frac{\partial \phi_i}{\partial p_j} \left(\frac{\partial p_j^*}{\partial x_i} \right) = 0. \tag{88}$$

Strategic effect. The strategic effect is

$$\psi(\mathbf{x}) \equiv (n-1) \frac{\partial}{\partial p_j} \phi_i(\mathbf{p}^*(\mathbf{x}), \mathbf{x}) \frac{\partial}{\partial x_i} p_j^*(\mathbf{x}). \tag{89}$$

Next we show that $\partial \phi_i / \partial p_j$ is strictly positive for $\lambda < 1$. We then show that $\partial p_j^* / \partial x_i < 0$ with strategic complements price competition and β high enough, and as a result the strategic effect is negative.

We can write the FOC with respect to R&D as

$$\frac{\partial}{\partial x_i}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda)+(n-1)\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda)\frac{\partial}{\partial x_i}p_j^*(\mathbf{x})=0,$$

and

$$\begin{aligned}\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda) &= (p^*-c(Bx))\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}+\lambda\left[q^*+(p^*-c(Bx))\frac{\partial D_j(\mathbf{p}^*)}{\partial p_j}\right]\\ &\quad +\lambda(n-2)(p^*-c(Bx))\frac{\partial D_k(\mathbf{p}^*)}{\partial p_j},\end{aligned}$$

which can be rewritten as

$$\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda)=\frac{-q^*}{v_\lambda}\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}+\lambda\frac{\partial D_j(\mathbf{p}^*)}{\partial p_j}+\lambda(n-2)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_j}\right]+\lambda q^*, \tag{90}$$

where we have used the FOC: $(p^*-c(Bx))=-q^*/v_\lambda$. To show that $\partial\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda)/\partial p_j>0$, we rewrite (90) as follows:

$$\begin{aligned}\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda) &= \frac{-q^*}{v_\lambda}\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}+\lambda\frac{\partial D_j(\mathbf{p}^*)}{\partial p_j}+\lambda(n-2)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_j}-\lambda v_\lambda\right]\\ &= \frac{-q^*}{v_\lambda}\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}+\lambda\frac{\partial D_j(\mathbf{p}^*)}{\partial p_j}+\lambda(n-2)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_j}\right.\\ &\quad \left.-\lambda\left(\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i}+\lambda(n-1)\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right)\right].\end{aligned}$$

Using now that in the symmetric equilibrium $\partial D_i/\partial p_i=\partial D_j/\partial p_j$ and $\partial D_i/\partial p_j=\partial D_k/\partial p_j=\partial D_k/\partial p_i$ for $i\neq j$, $j\neq k$, $i\neq k$ we can rewrite the above expression as follows

$$\begin{aligned}\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}),\mathbf{x},\lambda) &= \frac{-q^*}{v_\lambda}\left[1+\lambda\,(n-2)-\lambda^2\,(n-1)\right]\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}\\ &= \frac{-q^*}{v_\lambda}\,(1-\lambda)\,\Lambda\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}>0 \text{ for } \lambda<1.\end{aligned} \tag{91}$$

We now show that $\partial^2\phi_i/\partial x_j\partial p_i$ is negative or positive depending on whether β is high or low. Note that:

$$\frac{\partial^2\phi_i}{\partial x_i\partial p_i}(\mathbf{x})=-c'(Bx)\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i}+\lambda\,(n-1)\,\beta\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right]$$

and

$$\begin{aligned}\frac{\partial^2 \phi_i}{\partial x_j \partial p_i}(\mathbf{x}) &= -c'(Bx)\left[\beta\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda\frac{\partial D_j(\mathbf{p}^*)}{\partial p_i} + \lambda\,(n-2)\,\beta\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right] \\ &= -c'(Bx)\left\{\beta\left[\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \lambda(n-1)\beta\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right] + (1-\beta)\lambda\frac{\partial D_k(\mathbf{p}^*)}{\partial p_i}\right\}.\end{aligned}$$

Therefore, $\partial^2\phi_i/\partial x_j\partial p_i < 0$ for β high enough. From (36), we have

$$\frac{\partial}{\partial x_i} p_j^*(\mathbf{x}) = \frac{1}{\Omega}\left(\partial_{x_i p_i}\phi_i \partial_{p_i p_j}\phi_i - \partial_{x_h p_i}\phi_i \partial_{p_i p_i}\phi_i\right),$$

where in the symmetric equilibrium, and using that $p^* - c(Bx) = -q^*/v_\lambda$,

$$\partial_{p_i p_i}\phi_i(\mathbf{x}) = 2\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i} + \left(\frac{-q^*}{v_\lambda}\right)\left[\frac{\partial^2 D_i(\mathbf{p}^*)}{(\partial p_i)^2} + \lambda(n-1)\frac{\partial^2 D_k(\mathbf{p}^*)}{(\partial p_i)^2}\right]$$

and

$$\partial_{p_i p_j}\phi_i(\mathbf{x}) = \frac{\partial D_i(\mathbf{p}^*)}{\partial p_j} + \left(\frac{-q^*}{v_\lambda}\right)\left[(1+\lambda)\frac{\partial^2 D_i(\mathbf{p}^*)}{\partial p_j\partial p_i} + \lambda(n-2)\frac{\partial^2 D_k(\mathbf{p}^*)}{\partial p_j \partial p_i}\right] + \lambda\frac{\partial D_j(\mathbf{p}^*)}{\partial p_i}. \quad (92)$$

Strategic complements price competition $\partial_{p_i p_j}\phi_i(\mathbf{x}) > 0$, together with the assumption $\Delta_p < 0$, both imply that $\Omega > 0$. Note also that the assumption $v < 0$ implies $\partial^2\phi_i/\partial x_i\partial p_i < 0$, and since the expression for $\partial^2\phi_i/\partial x_j\partial p_i$ becomes negative for β high enough, we can establish:

$$\frac{\partial p_j^*}{\partial x_i} < 0 \text{ with strategic complements price competition and } \beta \text{ high enough,}$$

in which case the strategic effect is negative and firms adopt a "puppy dog" strategy (Fudenberg and Tirole 1984): increasing x_i decreases the prices of rivals because a larger x_i shifts the price best reply of firm j inwards as $\partial^2\phi_j/\partial x_i\partial p_j < 0$ and also shift inwards the price best reply of firm i since $\partial^2\phi_i/\partial x_i\partial p_i < 0$. The result is that the strategic effect is negative ($\psi < 0$) and we have puppy dog investment incentives.

Welfare. From our previous analysis:

$$W'(\lambda) = (p^* - c(Bx^*))n\frac{\partial q^*}{\partial\lambda} - \left(nc'(Bx^*)Bq^* + n\Gamma'(x^*)\right)\frac{\partial x^*}{\partial\lambda}.$$

The FOC with respect to x is

$$\Gamma'(x^*) = -c'(Bx^*)\left[\tau + (n-1)\omega(\lambda)\left(\tilde{\beta}(\lambda) - \beta\right)\right]q^*. \quad (93)$$

Inserting the FOCs $p^* - c(Bx^*) = -q^*/v_\lambda$ and (93) into the expression for $W'(\lambda)$ we obtain:

$$W'(\lambda) = \left\{ -\frac{1}{v_\lambda}\frac{\partial q^*}{\partial \lambda} - \left[(1-\lambda)\beta - \omega(\lambda)\left(\tilde{\beta}(\lambda) - \beta\right)\right](n-1)\, c'(Bx^*)\frac{\partial x^*}{\partial \lambda} \right\} nq^*. \qquad (94)$$

In Cournot when the strategic effect is negative (i.e.,$\left(\tilde{\beta}(\lambda) - \beta\right) < 0$) , the sign of the impact of λ on welfare in each region (R_{I}, R_{II} and R_{III}) is the same in the simultaneous and the two-stage model. This is the case also with Bertrand competition and β high (puppy dog strategy).

B.4 Model specifications

In this section we characterize the model with linear and constant elasticity demands analogs to AJ and CE. For each case, we first consider the simultaneous and then the two-stage model.

B.4.1 Linear model

Model specification: main assumptions. We assume the following: $D_i(p) = a - bp_i + m\sum_{j\neq i} p_j$ with $a, b, m > 0$; this linear direct demand obtains from a representative consumer with the following symmetric and strictly concave quadratic utility function:

$$U(q) = u_1 \sum_{i=1}^{n} q_i - \frac{1}{2}\left(u_2 \sum_{i=1}^{n} q_i^2 + 2u_3 \sum_{j\neq i} q_i q_j \right),$$

with $u_2 > u_3 > 0$, $u_1 > 0$, and where

$$a = \frac{u_1}{u_2 + (n-1)\, u_3},$$

$$b = \frac{u_2 + (n-2)u_3}{[u_2 + (n-1)\, u_3]\,(u_2 - u_3)}$$

and

$$m = \frac{u_3}{[u_2 + (n-1)\, u_3]\,(u_2 - u_3)}.$$

(See Vives 1999, pp. 146-147.)

The innovation function of firm i is $c_i = \bar{c} - x_i - \beta \sum_{j\neq i} x_j$ and the cost of investing x in R&D is given by $\Gamma(x) = (\gamma/2)x^2$. Linear demand satisfies Assumption 1B, the innovation and investment functions satisfy Assumptions A.2 and A.3. Under this model specification, we have $v = -b + (n-1)m$, and $v_\lambda = -b + \lambda(n-1)m$. According to the above analysis, we impose: $v < 0$, i.e. $b > (n-1)m$.

Simultaneous model. Interior equilibrium. By solving the FOCs and using that in the symmetric equilibrium $q^* = a + vp^*$, we derive the symmetric interior equilibrium:

$$p^* = \frac{v_\lambda (Ba\tau - \bar{c}\gamma) + a\gamma}{\Delta}$$

and

$$x^* = \frac{\tau(-v_\lambda)(\bar{c}v + a)}{\Delta}.$$

Second-order, stability and regularity conditions. It is straightforward to obtain that

$$\Delta_x = -\gamma,\ \Delta_p = v + v_\lambda = -2b + (n-1)m(1+\lambda), \text{ and } \Delta = -(v+v_\lambda)\gamma - vv_\lambda B\tau.$$

Because demand is linear, the regularity condition $\Delta_p < 0$ is implied by the assumption $v < 0$. We thus only have to impose the second regularity condition (64), therefore we assume $-(v+v_\lambda)\gamma > vv_\lambda B\tau$. Second order conditions are: $\partial_{p_ip_i}\phi_i = -2b < 0$, $\partial_{x_ix_i}\phi_i = -\gamma < 0$ and $\partial_{p_ip_i}\phi_i(\partial_{x_ix_i}\phi_i) - (\partial_{x_ip_i}\phi_i)^2 > 0$, which is equivalent to $2\gamma b > [-b + \lambda(n-1)\beta m]^2$.

Table B1: Linear Bertrand Model

Demand	$D_i(p) = a - bp_i + m\sum_{j\neq i} p_j$
$c_i =$	$\bar{c} - x_i - \beta\sum_{j\neq i} x_j$
$\Gamma(x) =$	$(\gamma/2)x^2$
$v =$	$-b + (n-1)m$
$v_\lambda =$	$-b + \lambda(n-1)m$
S.O.C	$\gamma b > [-b + \lambda(n-1)\beta m]^2/2$
Regularity Condition	$[-(v+v_\lambda)/vv_\lambda]\gamma > B\tau$

Comparative statics on λ and spillover thresholds. Recall that only R_{I} exits if $-\Delta_p < (\partial D_k(p^*)/\partial p_i)\Lambda(v/v_\lambda)$, i.e., if

$$-(v+v_\lambda) < m\Lambda\left(\frac{v}{v_\lambda}\right), \tag{95}$$

otherwise we can identify R_{II} and R_{III} by deriving the corresponding spillover threshold. From (73):

$$\text{sign}\left\{\frac{\partial x^*}{\partial\lambda}\right\} = \text{sign}\left\{\beta\left[-(v+v_\lambda)\frac{v_\lambda}{v} - \lambda(n-1)m\right] - m\right\}.$$

Therefore,

$$\text{if } \beta \leq \underline{\beta}(\lambda) \equiv \frac{m[b-(n-1)m]}{\lambda(n-1)^2(\lambda+2)m^2 - 4b(\lambda+1/4)(n-1)m + 2b^2}, \tag{96}$$

then $\partial x^*/\partial\lambda \leq 0$ and $\partial p^*/\partial\lambda > 0$ (R_{I}). It is easy to see that $\underline{\beta}(\lambda)$ depends only on m/b and that it is hump-shaped in m/b (with $\underline{\beta}(\lambda) = 0$ for $m/b = 0$ and for $m/b = 1/(n-1)$).

Note that $\chi = 0$, $y = 1$ and $\xi = \frac{\beta[b-\lambda(n-1)m]^2}{m\gamma}$, so

$$H = \frac{m\gamma}{[b-\lambda(n-1)m]^2}. \tag{97}$$

Since $v_\lambda < 0$, H is strictly increasing in λ. Thus,

$$\operatorname{sign}\left\{\frac{\partial p^*}{\partial\lambda}\right\} = \operatorname{sign}\{H - \beta B\}.$$

It follows that:

$$\beta' = \frac{-1+\sqrt{4H(n-1)+1}}{2(n-1)}. \tag{98}$$

As H is strictly increasing in ■, so is β' (in AJ and KMZ β' is independent of λ).

Figures B1a and B1b depict the spillover thresholds and the three regions. The threshold for R_{I} and R_{II} is given by (96), whereas the threshold for R_{II} and R_{III} is given by (98). For illustrative purposes we consider two cases that only differ in the number of firms. In Figure B1a, $n = 8$, and condition (95) is not satisfied for any value of λ if β is sufficiently high, and consequently R_{II} and/or R_{III} exist. In Figure B1b, $n = 10$, and condition (95) holds for $\lambda > 0.882$. Thus, for λ sufficiently high, only R_{I} exists irrespective of the spillover level.

Spillover thresholds and regions R_{I}, R_{II} and R_{III}[8]

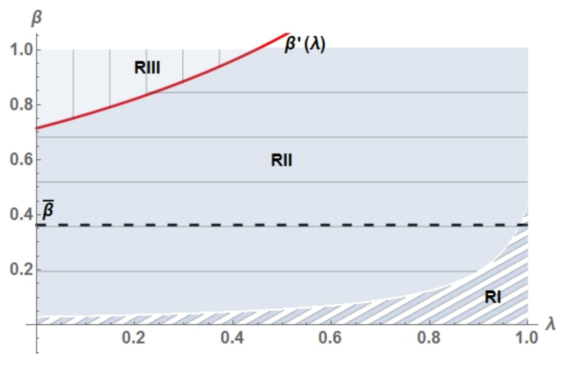

Fig. B1a. Linear Bertrand model.

($n = 8$)

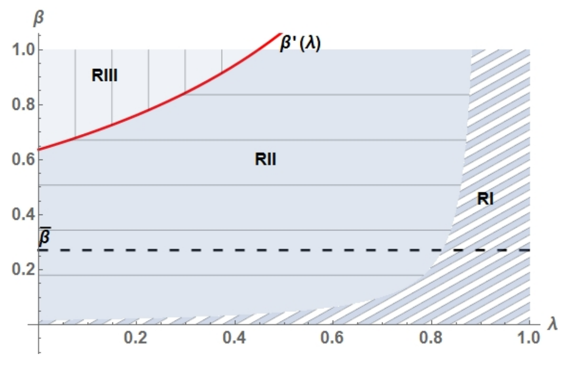

Fig. B1b. Linear Bertrand model.

($n = 10$)

Comparative statics on β'. Straightforward calculations show that the threshold $\beta'(\lambda)$ is strictly decreasing in b and strictly increasing in γ. These results are in line with the Cournot model.[9] We also obtain that $\beta'(\lambda)$ is strictly increasing in the slope of the direct demand

[8] All simulations are conducted for $a = 700$, $\bar{c} = 600$, $b = 1.4$, $m = 0.12$ and $\gamma = 70$.

[9] In Cournot $\beta'(0)$ is strictly increasing in b (see Table A3); recall that b is the (absolute value of the) slope of

with respect to rival prices, m. This follows since H has the same properties. However, H is increasing in u_3/u_2 for u_3/u_2 low (local monopolies) and decreasing in u_3/u_2 for u_3/u_2 close to 1 (homogenous products). Therefore $\beta'(\lambda)$ is non-monotone in u_3/u_2. It can also be showed that as in AJ and the CE model, $\beta'(0)$ is strictly decreasing in n (the threshold does not depend on n in KMZ), and therefore in terms of consumer surplus it is optimal to suppress overlapping ownership for any level of spillovers when firm entry is insufficient. In particular, this is the case in Bertrand with linear demand when $n < m\gamma/b^2$, in which case $\beta'(0) > 1$. More generally, the sign of $\partial\beta'(\lambda)/\partial n$ for some $\lambda \in (0,1)$ depends on the level of λ and n. Numerical simulations show that for low or moderate values of λ, $\partial\beta'(\lambda)/\partial n < 0$, whereas for high λ, $\partial\beta'(\lambda)/\partial n > 0$ if n is sufficiently high.[10]

PROPOSITION BL1 *Under the linear demand specification, if* $-(v + v_\lambda) < m\Lambda(v/v_\lambda)$ *then only region* R_{I} *exists. Otherwise, assume* $n > H(1)$*, where* H *is given by (97), and let* $\underline{\beta}(\lambda)$ *and* β' *be given, respectively, by (96) and (98). Then the following statements hold*:

(i) *if* $\beta \leq \underline{\beta}(\lambda)$, *then* $\frac{\partial q^*}{\partial\lambda} < 0$ *and* $\frac{\partial x^*}{\partial\lambda} \leq 0$ (R_{I});

(ii) *if* $\underline{\beta}(\lambda) < \beta \leq \beta'(\lambda)$, *then* $\frac{\partial q^*}{\partial\lambda} \leq 0$ *and* $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{II});

(iii) *if* $\beta > \beta'(\lambda)$, *then* $\frac{\partial q^*}{\partial\lambda} > 0$ *and* $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{III}).

We have that both $\underline{\beta}(\lambda)$ *and* $\beta'(\lambda)$ *are increasing in* λ *and hump-shaped in* u_3/u_2*, and* $\partial\beta'(0)/\partial n < 0$.

Profit. Simulations show that also in Bertrand with differentiated products and linear demand, profit in equilibrium is strictly increasing in the degree of overlapping ownership: $\pi^{*\prime}(\lambda) > 0$.

Welfare. First, we derive the threshold, $\bar{\beta}$, above which welfare increases with λ starting from $\lambda = 0$. We obtain $\bar{\beta}$ from the condition $W'(0) > 0$. Using (80), we only have to solve $F|_{\lambda=0} = 0$ for β to obtain the expression for $\bar{\beta}$. In particular, we have to solve

$$-\frac{\beta B v}{m} + \frac{v}{{v_\lambda}^2}\gamma - \beta(n-1)\left[\frac{\beta(-b+v)}{m} + \frac{v}{v_\lambda}\right] = 0,$$

or, equivalently,

$${v_\lambda}^2(n-1)(b-2v)\beta^2 - v_\lambda v\left[v_\lambda + m(n-1)\right]\beta + v\gamma m = 0.$$

the inverse demand in Cournot, while it is the slope of the direct demand with respect to own price in Bertrand.

[10]For example, for $b = 1.5$, $m = 0.1$ and $\gamma = 60$, $\partial\beta'(\lambda)/\partial n < 0$ for $n = 2.5$, but $\partial\beta'(\lambda)/\partial n > 0$ for $n = 6$ and $\lambda > 0.87$.

The above equation has two roots, only one of them can be positive since the denominator of the roots is $-2b(n-1)(b-2v) < 0$. Thus, $\bar{\beta}$ is given by

$$\bar{\beta} = \frac{v^2 - \sqrt{v\left\{v(n-1)^2m^2 + 2\left[(4\gamma - b)\,v - 2b\gamma\right](n-1)m + b^2v\right\}}}{-2b(n-1)(b-2v)}.$$

Numerical simulations confirm that the spillover thresholds satisfy $\beta'(0) > \bar{\beta}$.

Table B2: H and Spillover Thresholds in Linear Bertrand Model

$H =$	$m\gamma / \left[b - \lambda(n-1)m\right]^2$
$\underline{\beta}\,(\lambda) =$	$m\left[b-(n-1)m\right] / \left[\lambda(n-1)^2(\lambda+2)m^2 - 4b\,(\lambda+1/4)\,(n-1)m + 2b^2\right]$
$\bar{\beta} =$	$\left(v^2 - \sqrt{v\left\{v(n-1)^2m^2 + 2\left[(4\gamma-b)\,v - 2b\gamma\right](n-1)m + b^2v\right\}}\right) / \left[-2b(n-1)(b-2v)\right]$
$\beta' =$	$\left(-1 + \sqrt{4H(n-1)+1}\right) / \left[2(n-1)\right]$

Comparative statics on $\bar{\beta}$. As in Cournot (in AJ, KMZ and CE), the threshold $\bar{\beta}$ decreases with n. Similarly and in line with Cournot: $\bar{\beta}$ decreases with the slope of demand and increases with the parameter of the slope for the investment cost, γ. Regarding product differentiation: $\bar{\beta}$ is hump-shaped in u_3/u_2 since $\bar{\beta} = 0$ both for $u_3/u_2 = 0$ and $u_3/u_2 = 1$. Finally, also in Bertrand $\bar{\beta}$ may take values greater than 1 (so $\lambda^o_{\mathrm{TS}} = 0$ irrespective of the value of β) when there are a few firms in the market and γ (b) are sufficiently high (low). Note that in Figures B2a-c we assume that parameters a, b and m are fixed as n changes. This implies that parameters u_1, u_2 and u_3 must change with n (see Section B.4.1). Alternatively, we assume in Figure B2d that parameters u_1, u_2 and u_3 are fixed (such that $a = 750$, $b = 1.5$ and $m = 0.1$ for $n = 8$), while a, b and m change with n. Results are qualitatively the same: the thresholds in B2a and B2d are almost the same for n equal or close to 8, while they are higher in B2d than in B2a for two or three firms in the market.

Threshold value $\bar{\beta}$

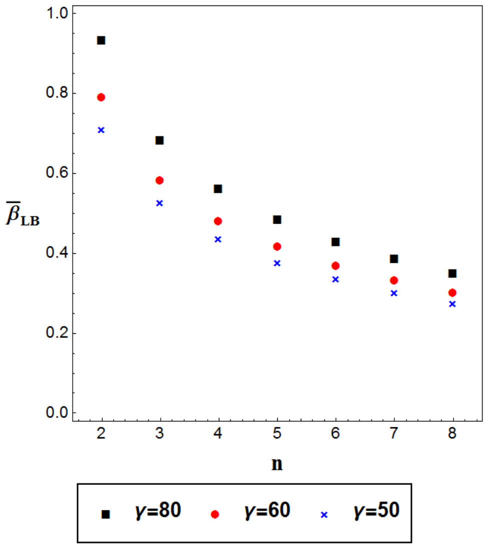

Fig. B2a. Linear Bertrand model.

($b = 1.5$, $m = 0.1$)

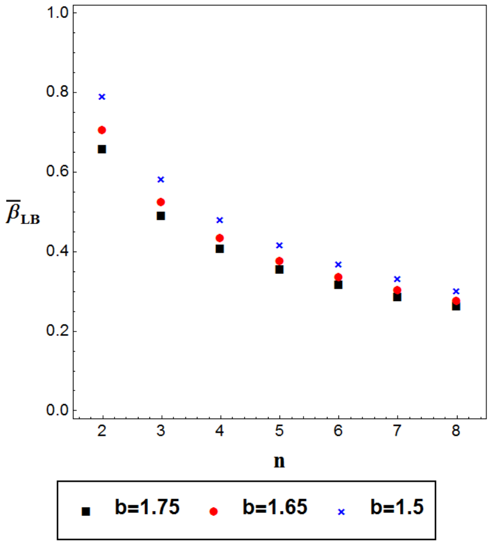

Fig. B2b. Linear Bertrand model.

($\gamma = 60$, $m = 0.1$)

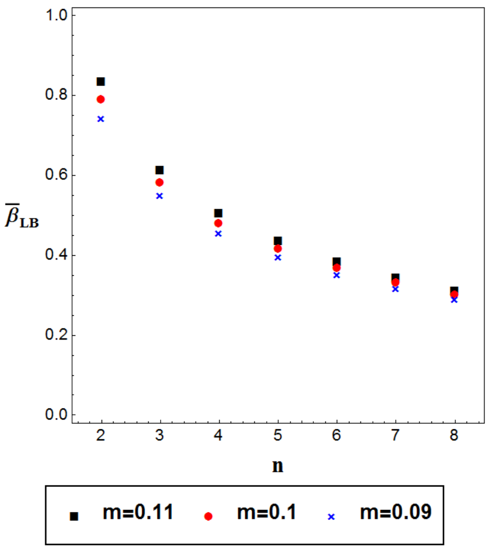

Fig. B2c. Linear Bertrand model.

($b = 1.5$, $\gamma = 60$)

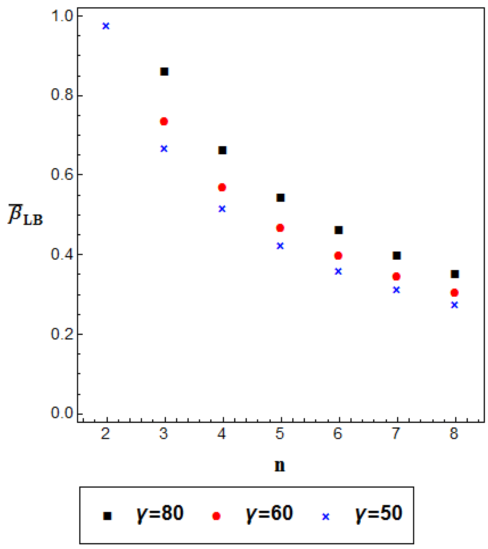

Fig. B2d. Linear Bertrand model.

($u_1 = 937.5$, $u_2 = 0.7$ and $u_3 = 0.078$)

Optimal degree of overlapping ownership (TS and CS standard)[11]

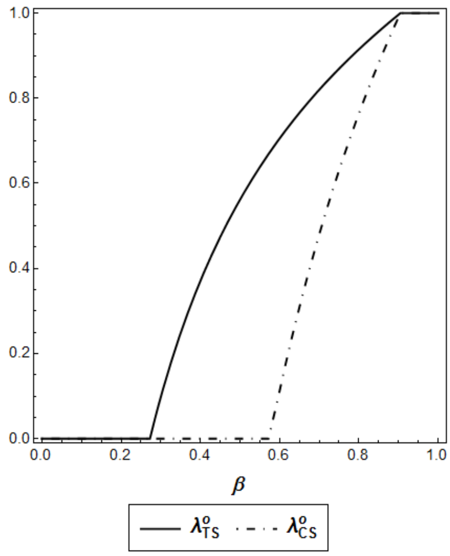

Fig. B3a. Linear Bertrand model.

($\gamma = 50$, $n = 6$)

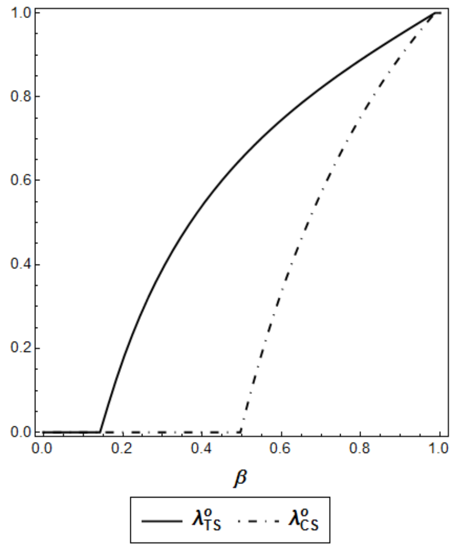

Fig. B3b. Linear Bertrand model.

($\gamma = 50$, $n = 8$)

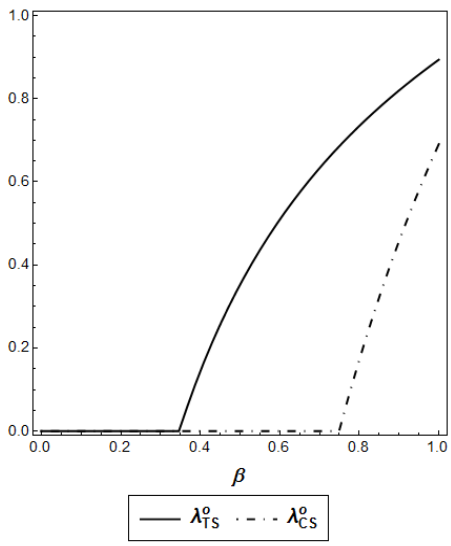

Fig. B3c. Linear Bertrand model.

($\gamma = 80$, $n = 6$)

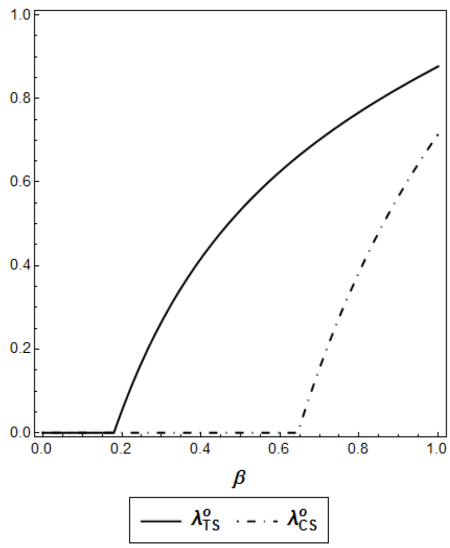

Fig. B3d. Linear Bertrand model.

($\gamma = 80$, $n = 8$)

Comparative statics on the socially optimal degree of overlapping ownership. Our simulations confirm that the main findings obtained in Cournot also hold in Bertrand; namely the socially optimal level of overlapping ownership increases with the size of spillovers and with the number of firms. Secondly, while the comparative statics are qualitatively similar in terms of consumer surplus, the scope for overlapping ownership is lower. Thirdly, Figures B3a-d show that for not

[11] All simulations are conducted for $a = 700$, $b = 1.5$, $m = 0.1$ and $\bar{c} = 500$.

too highly concentrated markets and high spillover levels, $\lambda = 1$ can be optimal in terms of total and consumer surplus. The thresholds $\bar{\beta}$ and $\beta'(0)$, as discussed above, decrease with n, and the optimal degrees of overlapping ownership λ^o_{TS} and λ^o_{CS}, decrease with the parameter of the slope for the investment cost, γ.

Optimal degree of overlapping ownership (TS and CS standard) [12]

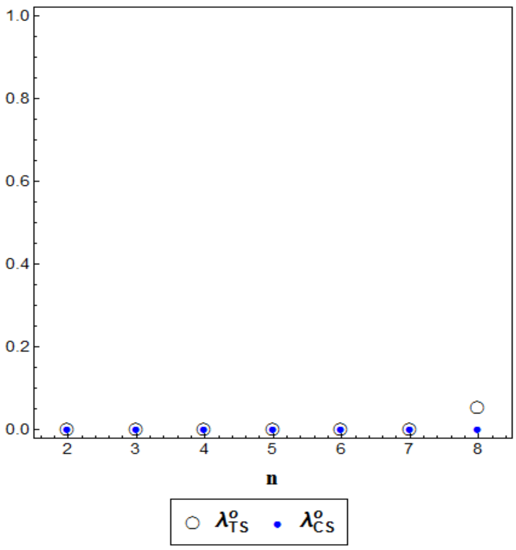

Fig. B4a. Linear Bertrand model.

($\gamma = 80$, $\beta = 0.2$)

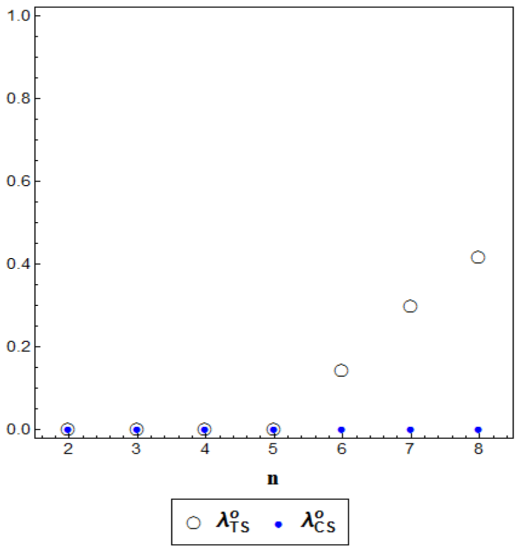

Fig. B4b. Linear Bertrand model.

($\gamma = 80$, $\beta = 0.4$)

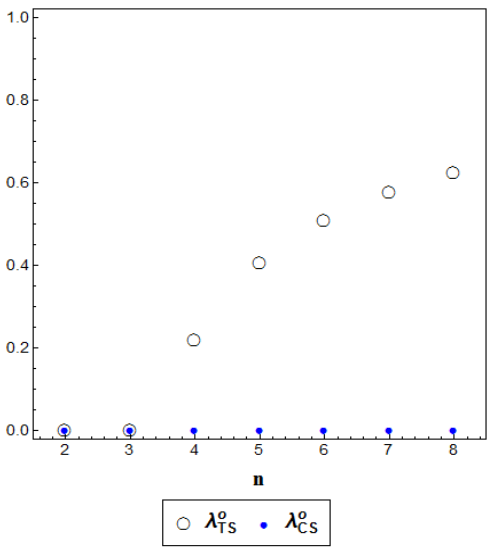

Fig. B4c. Linear Bertrand model.

($\gamma = 80$, $\beta = 0.6$)

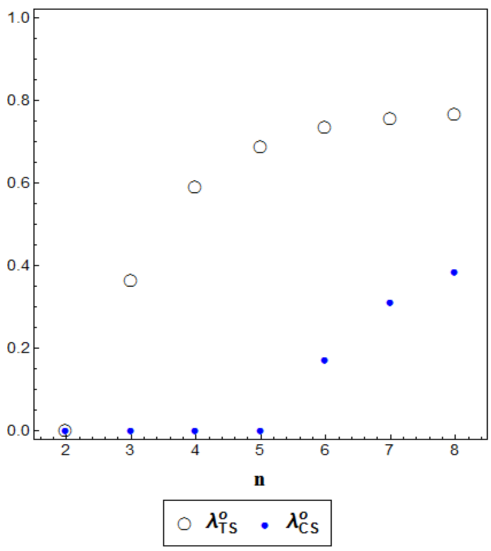

Fig. B4d. Linear Bertrand model.

($\gamma = 80$, $\beta = 0.8$)

[12] All simulations are conducted for $a = 750$, $b = 1.5$, $m = 0.1$ and $\bar{c} = 500$.

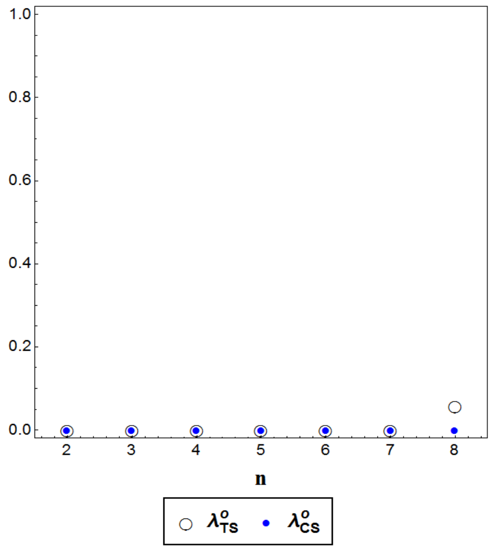

Fig. B4e. Linear Bertrand model.
($\gamma = 80$, $\beta = 0.2$, $u_1 = 937.5$, $u_2 = 0.7$
and $u_3 = 0.078$)

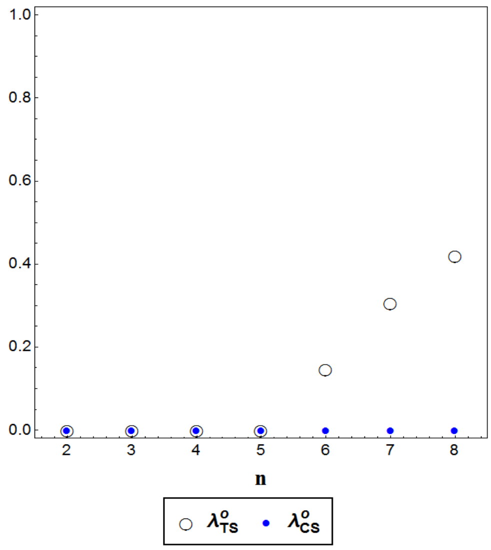

Fig. B4f. Linear Bertrand model.
($\gamma = 80$, $\beta = 0.4$, $u_1 = 937.5$, $u_2 = 0.7$
and $u_3 = 0.078$)

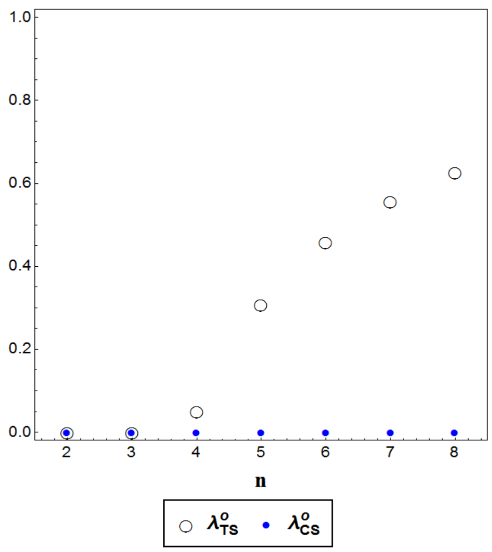

Fig. B4g. Linear Bertrand model.
($\gamma = 80$, $\beta = 0.6$, $u_1 = 937.5$, $u_2 = 0.7$
and $u_3 = 0.078$)

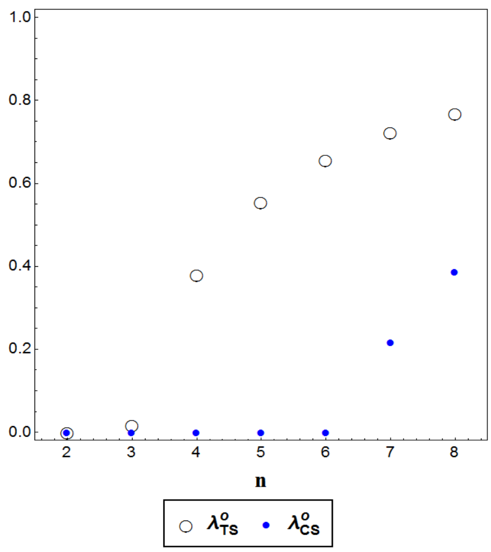

Fig. B4h. Linear Bertrand model.
($\gamma = 80$, $\beta = 0.8$, $u_1 = 937.5$, $u_2 = 0.7$
and $u_3 = 0.078$)

Finally, as Figures B4a-d indicate, it is not optimal to allow overlapping ownership for highly concentrated markets. As in the case of output competition, λ^o_{TS} increases weakly with the number of firms, and as in AJ and CE, λ^o_{CS} increases weakly with the number of firms and only if n is sufficiently large given the size of the spillover.[13] In Figures B4a-d we keep

[13] Recall that in KMZ the threshold β', and therefore sign$\{CS'(\lambda)\}$, are independent of the number of firms.

parameters a, b and m fixed as n changes, so parameters u_1, u_2 and u_3 must change with n. In Figures B4e-h, however, we allow parameters a, b and m to change with n by setting u_1, u_2 and u_3 at values such that $a = 750$, $b = 1.5$ and $m = 0.1$ for $n = 8$. Results are qualitatively the same in the two cases.

Optimal degree of overlapping ownership (TS and CS standard)[14]

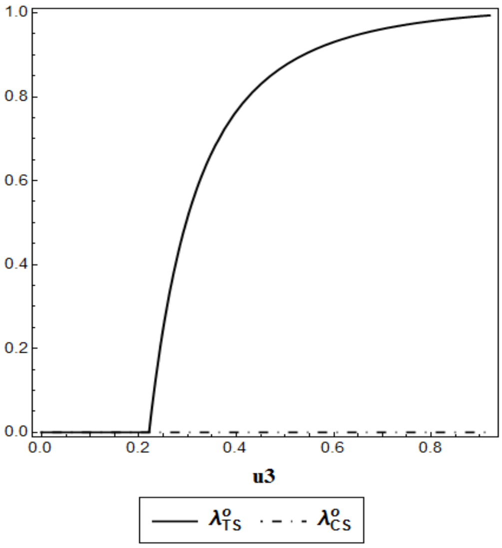

Fig. B5a. Linear Bertrand model.

($\beta = 0$, $\gamma = 150$, $n = 5$)

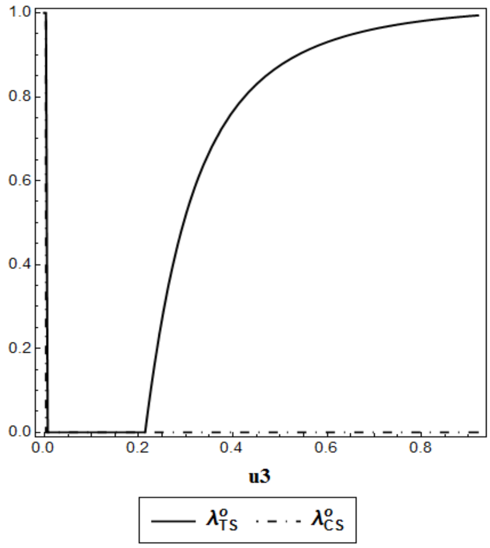

Fig. B5b. Linear Bertrand model.

($\beta = 0.25$, $\gamma = 150$, $n = 5$)

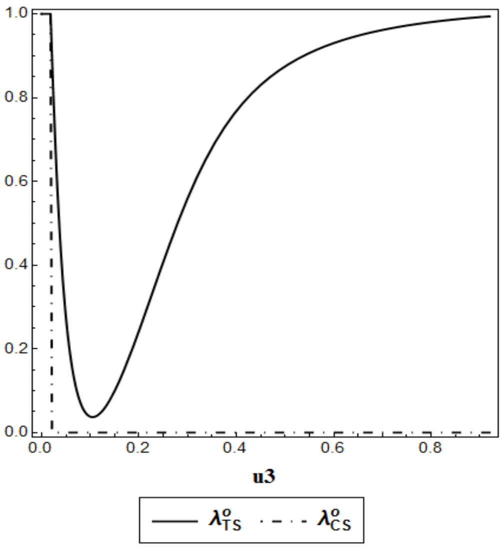

Fig. B5c. Linear Bertrand model.

($\beta = 0.75$, $\gamma = 150$, $n = 5$)

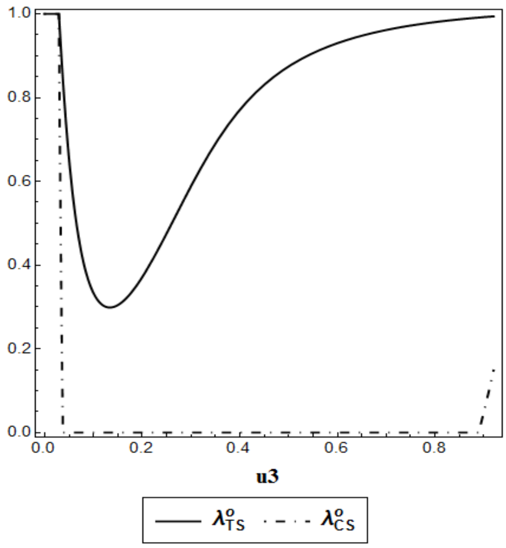

Fig. B5d. Linear Bertrand model.

($\beta = 1$, $\gamma = 150$, $n = 5$)

Comparative statics on the degree of product differentiation. Here, we fix $u_2 = 1$, and we then compute the optimal degrees of overlapping ownership (λ^o_{TS} and λ^o_{CS}) for values of u_3

[14] All simulations are conducted for $a = 700$ and $\bar{c} = 500$.

ranging from 0 (which reflects the monopoly case) to 0.92 (which reflects the case of intense competition because of very low product differentiation). To guarantee that the regularity condition is satisfied for $u_3 \in [0, 0.92]$ we consider $n = 5$ and $\gamma = 150$. Simulations show that for $\beta > 0$, λ^o_{TS} is U-shaped, and so is λ^o_{CS} is for β sufficiently high (see Figures B5a-d). For $\beta > 0$, if $u_3 \to 0$, then λ^o_{TS}, $\lambda^o_{\text{CS}} \to 1$. The U-shaped pattern is robust and also appears for higher/lower values of n and γ. In particular, in Figures B6a-b we conduct similar simulations but assuming $n = 8$ and $\gamma = 60$.

Optimal degree of overlapping ownership (TS and CS standard)[15]

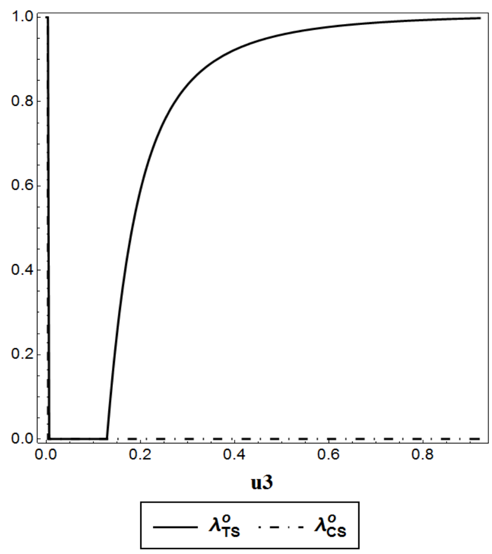

Fig. B6a. Linear Bertrand model.

($\beta = 0.1$, $\gamma = 60$, $n = 8$)

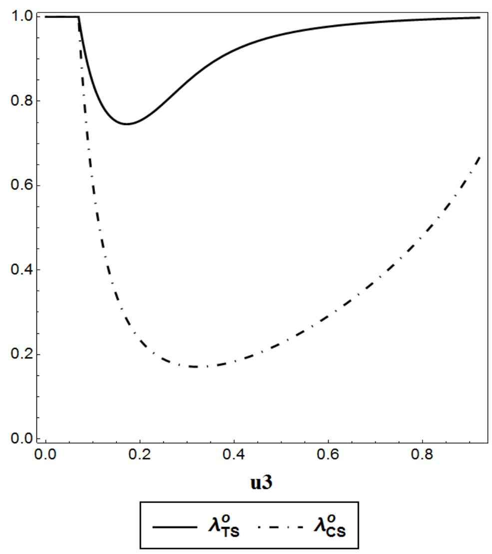

Fig. B6b. Linear Bertrand model.

($\beta = 0.9$, $\gamma = 60$, $n = 8$)

Two-stage model. Interior equilibrium. By solving the FOCs (87) and (88) with $c = \bar{c} - Bx^*$ and $q^* = a + vp^*$, we obtain the symmetric interior equilibrium:

$$p^* = \frac{\{\bar{c}\gamma - a\,[(n-1)s(\lambda) + \tau]\,B\}\,v_\lambda - a\gamma}{\{\gamma + v\,[(n-1)\,s(\lambda) + \tau]\,B\}\,v_\lambda + v\gamma}$$

and

$$x^* = \frac{v_\lambda(\bar{c}v + a)\,[(n-1)s(\lambda) + \tau]}{\{\gamma + v\,[(n-1)s(\lambda) + \tau)]\,B\}\,v_\lambda + v\gamma},$$

where $s(\lambda) \equiv \omega(\lambda)(\tilde{\beta}(\lambda) - \beta)$, and $\omega(\lambda)$ and $\tilde{\beta}(\lambda)$ are obtained below.

Strategic effect. Here, we first obtain $\partial p^*_j(\mathbf{x})/\partial x_i$, and we then derive the expressions for the strategic effect of investment (ψ) and the threshold $\tilde{\beta}(\lambda)$. With linear demand we have $\partial_{x_i p_i}\phi_i(\mathbf{x}) = -c'(Bx)\,[-b + \lambda\,(n-1)\,\beta m]$ and $\partial_{x_j p_i}\phi_i(\mathbf{x}) = -c'(Bx)\,[-b\beta + \lambda m + \lambda\,(n-2)\,\beta m]$.

[15] All simulations are conducted for $a = 700$ and $\bar{c} = 500$.

We also have that $\partial_{p_ip_i}\phi_i(\mathbf{x}) = -2b$ and $\partial_{p_ip_j}\phi_i(\mathbf{x}) = m\,(1+\lambda)$. Therefore,

$$\partial_{p_ip_i}\phi_i(\mathbf{x}) - \partial_{p_ip_j}\phi_i(\mathbf{x}) = -2b - m(1+\lambda),$$

$$\partial_{p_ip_i}\phi_i(\mathbf{x}) + (n-1)\partial_{p_ip_j}\phi_i(\mathbf{x}) = -2b + (n-1)m(1+\lambda),$$

$$\partial_{p_ip_j}\phi_i(\mathbf{x}) - \partial_{p_ip_i}\phi_i(\mathbf{x})\beta = m\,(1+\lambda) + 2b\beta.$$

Using (36) we can write

$$\frac{\partial}{\partial x_i} p_j^*(\mathbf{x}) = \frac{-c'(Bx)}{\Omega}\left[\varphi(\lambda)\beta - (1-\lambda)\,bm\right],$$

where $\Omega = [-2b - m(1+\lambda)]\,[-2b + (n-1)m(1+\lambda)] > 0$, and $\varphi(\lambda) = \lambda\,(1+\lambda)\,(n-1)\,m^2 + 2\lambda\,(n-2)\,bm - 2b^2 < 0$, since $\varphi(0) = -2b^2 < 0$, $\varphi(1) = 2(b+m)\,[-b+m(n-1)] < 0$ and $\varphi'(\lambda) > 0$. Therefore,

$$\frac{\partial}{\partial x_i} p_j^*(\mathbf{x}) < 0.$$

From (91) we may write

$$\begin{aligned} \frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}), \mathbf{x}, \lambda) &= \frac{-q^*}{v_\lambda}\left\{m\left[1 + \lambda\,(n-2)\right] - b\lambda\right\} + \lambda q^* \\ &= -\frac{q^*}{v_\lambda} m\,(1-\lambda)\,\Lambda. \end{aligned}$$

Note that $(1-\lambda)\Lambda$ is strictly positive for all $\lambda < 1$, thus, and as expected, for $\lambda < 1$:

$$\frac{\partial}{\partial p_j}\phi_i(\mathbf{p}^*(\mathbf{x}), \mathbf{x}, \lambda) > 0.$$

Therefore, the strategic effect of investment is

$$\begin{aligned} \psi &\equiv (n-1)\frac{\partial \phi_i}{\partial p_j}\left(\frac{\partial p_j^*}{\partial x_i}\right) \\ &= -(n-1)\frac{q^*}{v_\lambda} m\,(1-\lambda)\,\Lambda\left(\frac{-c'(Bx)}{\Omega}\left[\varphi(\lambda)\beta - (1-\lambda)\,bm\right]\right) \\ &= \frac{-c'(Bx)}{\Omega}\left(-\frac{q^*}{v_\lambda}\right) m(n-1)\,(1-\lambda)\,\Lambda\left[\varphi(\lambda)\beta - (1-\lambda)\,bm\right] < 0. \end{aligned}$$

We can rewrite the strategic effect of investment as

$$\psi = -c'(Bx)q^*\omega(\lambda)\left(\tilde{\beta}(\lambda) - \beta\right),$$

where

$$\omega(\lambda) = \frac{m(n-1)\,(1-\lambda)\,\Lambda\varphi(\lambda)}{\Omega v_\lambda} > 0 \text{ and } \tilde{\beta}(\lambda) = \frac{1-\lambda}{\varphi(\lambda)}bm < 0.$$

Welfare. The expression for $W'(\lambda)$ is given by (94). Recall that in Cournot only when the strategic effect is negative, the sign of the impact of λ on welfare in each region (R_{I}, R_{II} and R_{III}) is the same in the simultaneous and the two-stage model. The reason is that the factor that multiplies $\partial x^*/\partial\lambda$ in the expression for $W'(\lambda)$ is positive. When the strategic effect is positive and spillovers are low, the factor is negative and as a result, welfare decreases with λ in R_{II}, and can increase or decrease with λ in R_{I} and in R_{III}. In the Bertrand model with linear demand, the strategic effect is always negative, and as in Cournot, the factor that multiplies $\partial x^*/\partial\lambda$ is positive. (Note also that $-1/v_\lambda > 0$.) Therefore, the sign of the impact of λ on welfare in each region (R_{I}, R_{II} and R_{III}) is the same in the simultaneous and the two-stage model: $W'(\lambda) < 0$ when x^* decreases and p^* increases with λ (as in R_{I}), $W'(\lambda) > 0$ when x^* increases and p^* decreases with λ (as in R_{III}), and $W'(\lambda) \gtrless 0$ when x^* and p^* increase with λ (as in R_{II}).

The next figures depict the threshold $\bar{\beta}^{2S}_{LB}$ above which welfare increases with λ at $\lambda = 0$.

Threshold value $\bar{\beta}$[16]

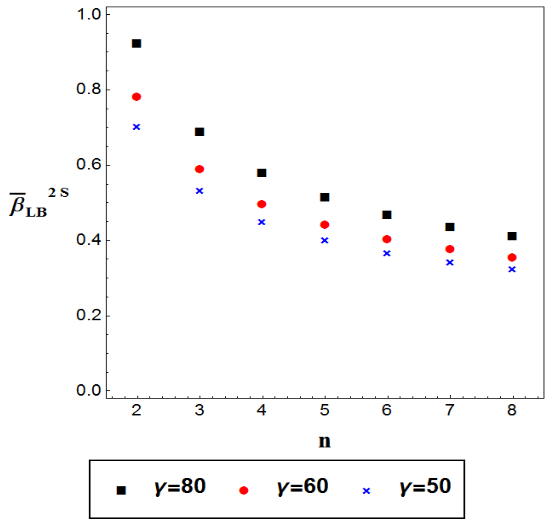

Fig. B7a. Linear Bertrand two-stage model. $(b = 1.5, m = 0.1)$

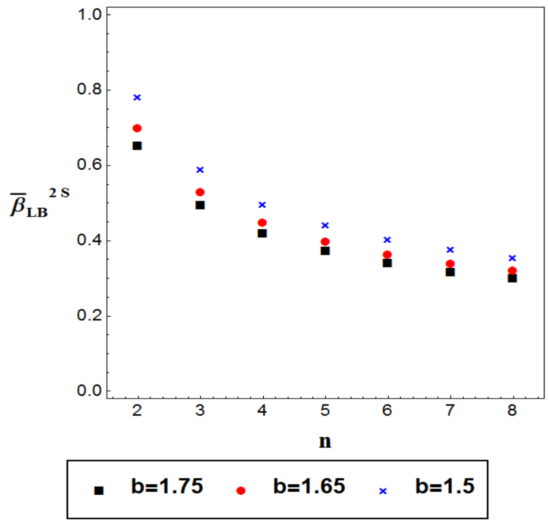

Fig. B7b. Linear Bertrand two-stage model. $(m = 0.1, \gamma = 60)$

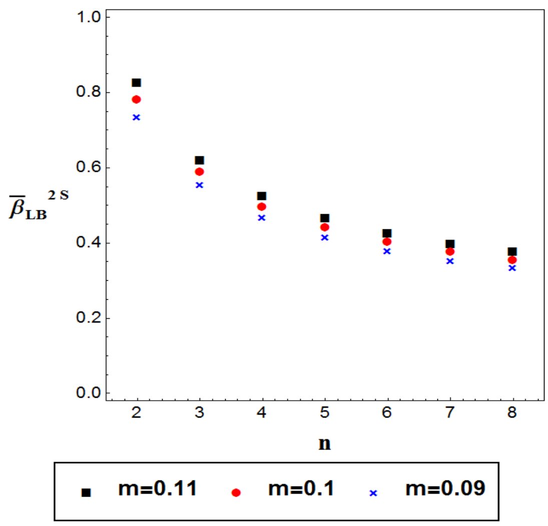

Fig. B7c. Linear Bertrand two-stage model. $(b = 1.5, \gamma = 60)$

Comparative statics on $\bar{\beta}_{LB}^{2S}$. Results are consistent with those obtained in Cournot and in simultaneous Bertrand: the threshold $\bar{\beta}_{LB}^{2S}$ increases with m and γ, and decreases with n and with b. In addition, and in line with the other models, $\bar{\beta}_{LB}^{2S}$ may be greater than 1 (and thus $\lambda_{\text{TS}}^{o} = 0$ for all β) when there are few firms in the market and γ (b) are sufficiently high (low).

[16] In the three simulations: $a = 900$ and $\bar{c} = 500$.

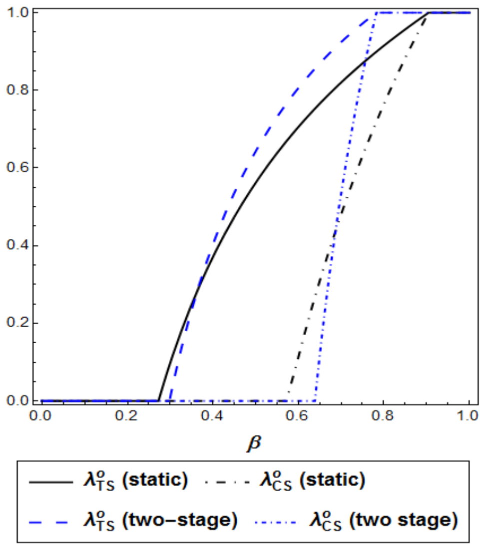

Fig. B8a. Linear Bertrand two-stage model. ($\gamma = 50$, $n = 6$)

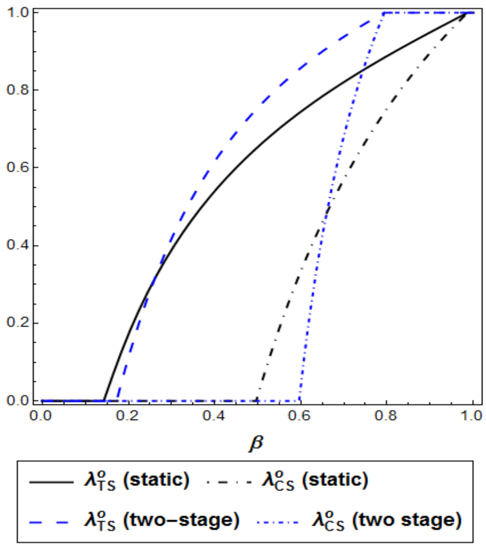

Fig. B8b. Linear Bertrand two-stage model. ($\gamma = 50$, $n = 8$)

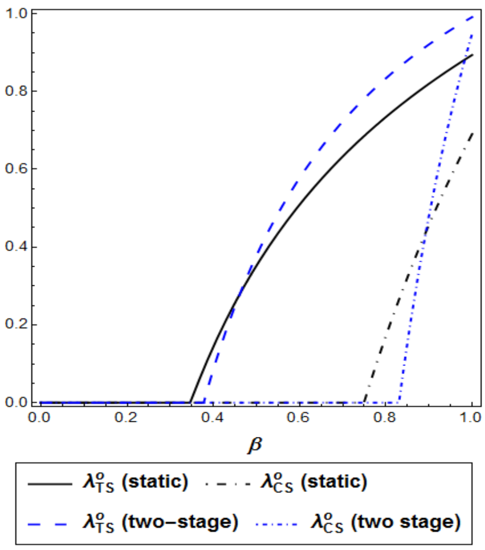

Fig. B8c. Linear Bertrand two-stage model. ($\gamma = 80$, $n = 6$)

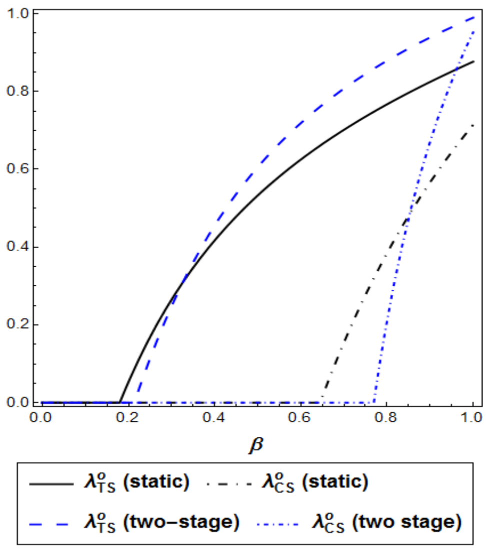

Fig. B8d. Linear Bertrand two-stage model. ($\gamma = 80$, $n = 8$)

Comparative statics on the socially optimal degree of overlapping ownership. Results are similar to those obtained in Cournot with two stages: λ^o_{TS} increases with β and n, and when R&D has commitment value λ^o_{TS} tends to be higher than in the simultaneous model when spillovers are high. However and unlike the Cournot model, we do not observe cases in which $\lambda^o_{\mathrm{CS}} > \lambda^o_{\mathrm{TS}}$. The reason is that those cases may arise in Cournot when the strategic effect is

[17] All simulations are conducted for $a = 900$, $b = 1.5$, $m = 0.1$ and $\bar{c} = 500$.

positive; in Bertrand with linear demand the strategic effect is always negative. Finally, in line with the simultaneous case, λ^o_{TS} and λ^o_{CS} decrease with γ. Note also that we do not have a bang-bang solution for CS.

Optimal degree of overlapping ownership (TS and CS standard)[18]

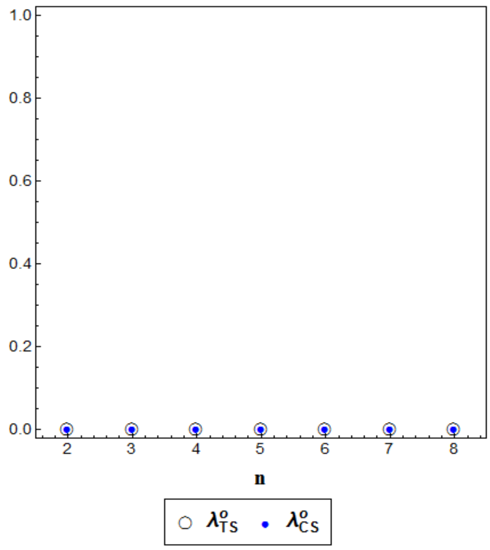

Fig. B9a. Linear Bertrand two-stage model. ($\gamma = 80$, $\beta = 0.2$)

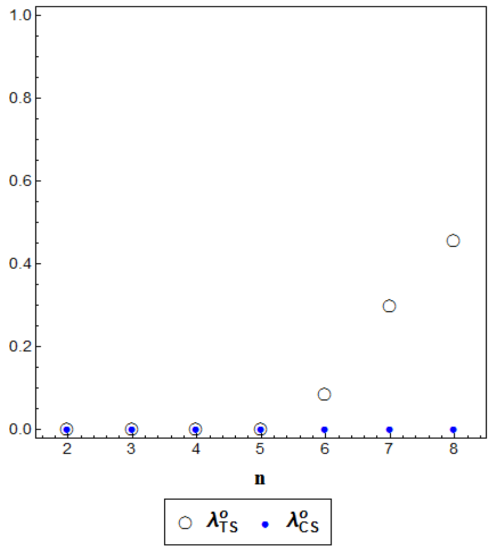

Fig. B9b. Linear Bertrand two-stage model. ($\gamma = 80$, $\beta = 0.4$)

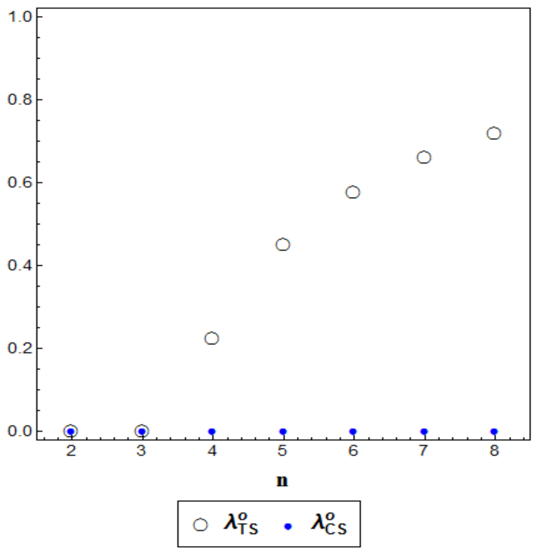

Fig. B9c. Linear Bertrand two-stage model. ($\gamma = 80$, $\beta = 0.6$)

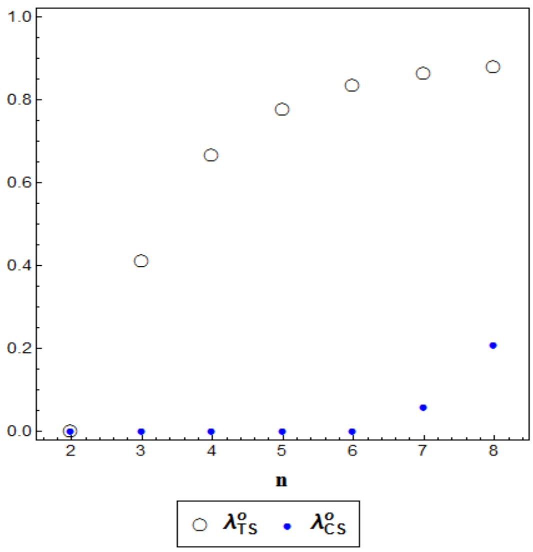

Fig. B9d. Linear Bertrand two-stage model. ($\gamma = 80$, $\beta = 0.8$)

Figures B9a-d confirm that it is not optimal to allow overlapping ownership for highly concentrated markets. In line with the other models, λ^o_{TS} weakly increases with the number of firms, and λ^o_{CS} increases weakly with n (only if n is sufficiently large given the size of the

[18] All simulations are conducted for $a = 900$, $b = 1.5$, $m = 0.1$ and $\bar{c} = 500$.

spillover).

Optimal degree of overlapping ownership (TS and CS standard)[19]

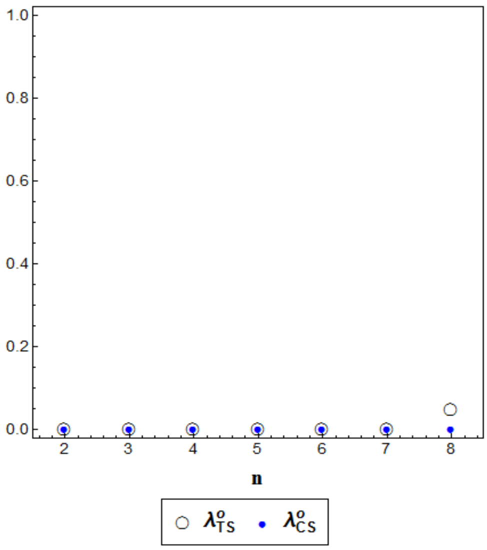

Fig. B10a. Linear Bertrand two-stage model. ($\gamma = 60$, $\beta = 0.2$)

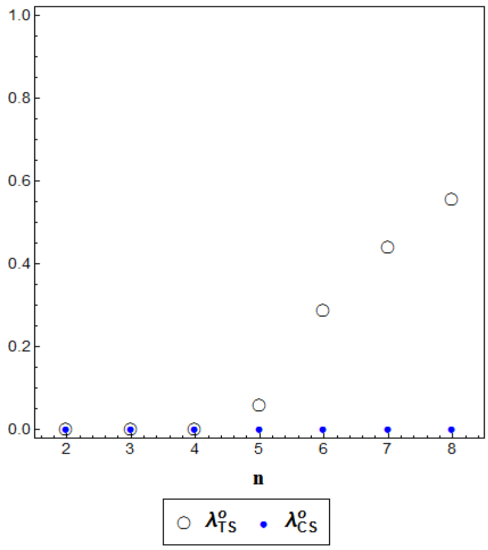

Fig. B10b. Linear Bertrand two-stage model. ($\gamma = 60$, $\beta = 0.4$)

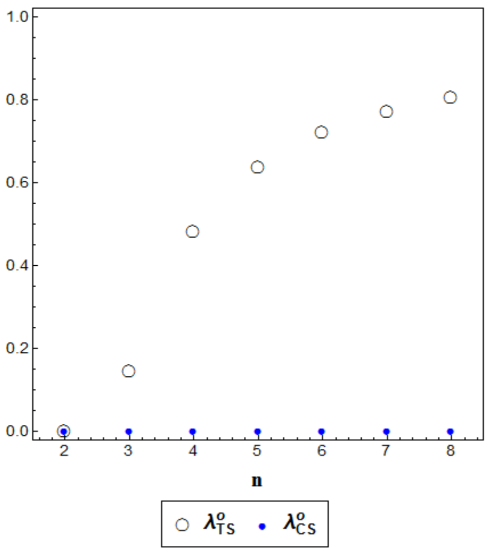

Fig. B10c. Linear Bertrand two-stage model. ($\gamma = 60$, $\beta = 0.6$)

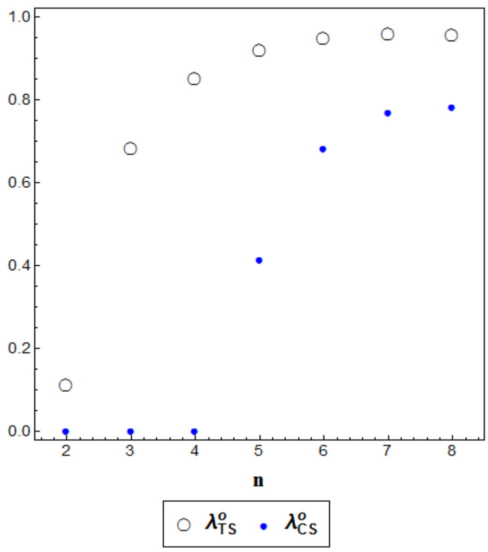

Fig. B10d. Linear Bertrand two-stage model. ($\gamma = 60$, $\beta = 0.8$)

[19] All simulations are conducted for $a = 900$, $b = 1.5$, $m = 0.1$ and $\bar{c} = 500$.

B.4.2 Constant elasticity model

Model specification: main assumptions. Consider the following form for the representative consumer's utility function

$$U = \left[\sum_{i=1}^{n} q_i^{\rho}\right]^{1/\rho} q_0^{\theta},$$

with $\rho \in (0,1)$ and $\theta > 0$, and where q_0 is the numéraire and q_i is quantity for the variety i of the differentiated product. The consumer's problem consists of maximizing U subject to the budget constraint $\sum_{i=0}^{n} p_i q_i = Y$, where Y is aggregate income. The demand functions resulting from this problem are

$$D_i(\mathbf{p}) = \frac{p_i^{-1-1/\mu}}{\sum_{j=1}^{n} p_j^{-1/\mu}} S,$$

where $\mu = (1-\rho)/\rho \in (0,\infty)$, and $S \equiv Y/(1+\theta)$ is the total spending on the differentiated product variants; the amount of numéraire is $q_0 = \theta S$. Note that $\sigma = 1/(1-\rho)$ is the constant elasticity of substitution between any two products. As $\rho \to 1$ ($\sigma \to \infty$), products become perfect substitutes, while as $\rho \to 0$ ($\sigma \to 1$), products become independent.

The innovation function is $c_i = \kappa(x_i + \beta \sum_{j \neq i} x_j)^{-\alpha}$ with α, $\kappa > 0$, whereas the investment cost function is $\Gamma(x_i) = x_i$. Thus, the innovation and investment functions satisfy Assumptions A.2 and A.3. CE demand, as specified, is not quasilinear, but it is smooth and downward sloping, the demand system is symmetric and products are gross substitutes (Assumption 1B). From Table B3, we get at the symmetric equilibrium

$$v^* = -\frac{S}{np^{*2}} < 0, \text{ and } v_{\lambda}^* = -\frac{(n-1)(1-\lambda)+\mu n}{n^2 p^{*2} \mu} S < 0.$$

Table B3: CE Demand Bertrand Basic Derivatives
for $i \neq j,\ j \neq k,\ i \neq k$

$\partial D_i(p^*)/\partial p_i =$	$-\frac{S}{n^2 p^{*2} \mu}(n-1+n\mu)$
$\partial^2 D_i(p^*)/\ \partial p_i{}^2 =$	$\frac{2S}{n^3 p^{*3} \mu^2}\left[n^2\mu^2 + \frac{3}{2}(n-1)n\mu + \frac{1}{2}(n-2)(n-1)\right]$
$\partial D_i(p^*)/\partial p_j =$	$\frac{S}{n^2 p^{*2} \mu}$
$\partial D_i(p^*)/\ \partial p_j{}^2 =$	$-\frac{S}{n^3 p^{*3} \mu^2}\left[(n-2)+n\mu\right]$
$\partial^2 D_i(p^*)/\partial p_j \partial p_i =$	$-\frac{S}{n^3 p^{*3} \mu^2}\left[(n-2)+n\mu\right]$
$\partial^2 D_i(p^*)/\partial p_k \partial p_j =$	$\frac{2S}{n^3 p^{*3} \mu^2}$

Simultaneous model. Interior equilibrium. The FOCs in the symmetric solution are given by (61) and (62):

$$\frac{p^* - c(Bx^*)}{p^*} = \frac{1}{\eta_i - \lambda(n-1)\eta_{ik}};$$

$$-c'(Bx^*)q^*\tau = \Gamma'(x^*),$$

where $\eta_i = (n-1+n\mu)/n\mu$ and $\eta_{ik} = 1/n\mu$. In the symmetric solution: $D_i = D_k = q^* = S/(np^*)$, $c_i = \kappa(Bx^*)^{-\alpha}$, $c' = \partial c_i/\partial x_i|_{x_i=x^*} = -\alpha\kappa(Bx^*)^{-\alpha-1}$, and $\Gamma'(x^*) = 1$; by solving the system of FOCs for p^* and x^* we get the symmetric interior equilibrium:

$$x^* = \frac{\alpha\tau SA}{Bn} \tag{99}$$

$$p^* = \frac{\kappa}{A\left(\alpha\tau SA/n\right)^{\alpha}}, \tag{100}$$

where

$$A = 1 + \frac{\mu n}{\Lambda - n(1+\mu)} = \frac{n-\Lambda}{n-\Lambda+n\mu} > 0 \text{ for } \lambda < 1.$$

Table B4: CE Bertrand Model

Demand	$D_i(p) = Sp_i^{-1-1/\mu}/\sum_{j=1}^{n} p_j^{-1/\mu}$
$c_i =$	$\kappa(x_i + \beta\sum_{j\neq i} x_j)^{-\alpha}$
$\Gamma(x) =$	x
$v =$	$-S/np^{*2}$
$v_\lambda =$	$-S\left[(n-1)(1-\lambda)+n\mu\right]/n^2p^{*2}\mu$
S.O.C	$\tau n^4\mu(n-\Lambda)\tilde{\lambda}(1+\alpha)(1+\mu) - n^2\alpha A\left(\mu n + n - \Lambda\right)\left[(1+\mu)\,n - \tau\right]^2 > 0$
Regularity Condition	$\lambda < 1$

with $\tilde{\lambda} = 1 + \lambda(n-1)\beta^2$.

Second-order, stability and regularity conditions. We first check the stability and regularity conditions; using (65) and (70) and from Table B3 we obtain

$$\Delta_x = -(1+\alpha)\frac{Bn}{\alpha\tau SA} < 0,$$

$$\Delta_p = -\frac{(1-\lambda)(n-1)}{n^2p^{*2}\mu}S < 0 \tag{101}$$

and

$$\begin{aligned} \Delta &= \Delta_p\Delta_x - \tau Bvv_\lambda\left(c'(Bx^*)\right)^2 \\ &= \frac{A\left(\frac{\alpha\tau SA}{n}\right)^{2\alpha}B}{n\kappa^2\mu\alpha\tau}(n-1)(1-\lambda) > 0 \end{aligned}$$

for $\lambda < 1$.

Second order conditions are: (i) $\partial_{p_ip_i}\phi_i < 0$; (ii) $\partial_{x_ix_i}\phi_i < 0$; and (iii) $\partial_{p_ip_i}\phi_i\left(\partial_{x_ix_i}\phi_i\right) -$

$(\partial_{p_i x_i}\phi_i)^2 > 0$. Conditions (i) and (ii) are satisfied:

$$\begin{aligned}
\partial_{p_i p_i}\phi_i &= 2\frac{\partial D_i(p^*)}{\partial p_i} - \left(\frac{q^*}{v_L}\right)\left[\frac{\partial^2 D_i(p^*)}{\partial p_i^2} + \lambda(n-1)\frac{\partial^2 D_k(p^*)}{\partial p_i^2}\right] \\
&= -\frac{S^{1+2\alpha}\left(\frac{\alpha\tau}{n}\right)^{2\alpha} A^{2\alpha+2}(1-\lambda)(1+\mu)\left(\frac{n-1}{n}\right)}{\left[(1-\lambda)(n-1)+\mu n\right]\mu\kappa^2} < 0
\end{aligned} \tag{102}$$

and

$$\begin{aligned}
\partial_{x_i x_i}\phi_i &= -c''(Bx^*)\left[1+\lambda(n-1)\beta^2\right]q^* - \Gamma''(x^*) \\
&= -\left(\frac{1+\alpha}{\alpha}\right)\frac{n}{\tau S A}\left[1+\lambda(n-1)\beta^2\right] < 0.
\end{aligned}$$

Using that

$$\partial_{p_i x_i}\phi_i = -\frac{\left(\frac{\alpha\tau S A}{n}\right)^{\alpha} A}{\kappa n\mu\tau}\left[(n-1)(1-\beta\lambda)+n\mu\right],$$

we have that condition (iii) is satisfied iff

$$-\frac{\left(\frac{\alpha\tau S A}{n}\right)^{2\alpha} A}{\mu^2 n^4\left[(n-1)(1-\lambda)+\mu n\right]\kappa^2\tau\alpha}\{D-E\} > 0$$

where $D \equiv A^3\left[(-\beta\lambda+\mu+1)n+\lambda\beta-1\right]^2\left[(n-1)(1-\lambda)+\mu n\right]\tau S^2\alpha^3\frac{n^2}{(\alpha\tau S A)^2}$ and $E \equiv n^4\mu(1-\lambda)(n-1)\left[1+\lambda(n-1)\beta^2\right](1+\alpha)(1+\mu)$. Therefore, the SOC reduces to

$$\tau n^4\mu(n-\Lambda)\tilde{\lambda}(1+\alpha)(1+\mu) - n^2\alpha A(\mu n+n-\Lambda)\left[(1+\mu)n-\tau\right]^2 > 0,$$

where $\tilde{\lambda} \equiv 1+\lambda(n-1)\beta^2$.

Comparative statics on λ and spillover thresholds. Recall that only R_{I} exits (irrespective of the spillover level) if $-\Delta_p < (\partial D_k(p^*)/\partial p_i)\,\Lambda(v/v_\lambda)$; replacing terms and simplifying the condition reduces to

$$\mu n(2\Lambda-n)-(n-1)^2(1-\lambda)^2 > 0, \tag{103}$$

which holds for $\lambda = 1$. If (103) does not hold, then we may identify R_{II} and R_{III} by deriving the corresponding spillover threshold. From (99) we have that

$$\frac{\partial x^*}{\partial\lambda} = \frac{2\alpha S(n-1)}{Bn\left[n-\Lambda+\mu n\right]^2}\left\{-\left[\left(-\frac{\lambda^2}{2}+(1+\mu)\lambda-\frac{1+\mu}{2}\right)n+\frac{(1-\lambda)^2}{2}\right]\beta(n-1)-\frac{\mu n}{2}\right\},$$

which implies that

$$\operatorname{sign}\left\{\frac{\partial x^*}{\partial\lambda}\right\} = \operatorname{sign}\left\{\beta\left[\left(\frac{\lambda^2}{2}-(1+\mu)\lambda+\frac{1+\mu}{2}\right)n-\frac{(1-\lambda)^2}{2}\right](n-1)-\frac{\mu n}{2}\right\}.$$

Therefore,

$$\text{if } \beta \leq \underline{\beta}(\lambda) \equiv \frac{\mu n}{(n-1)\left\{\left[\lambda^2 + (1+\mu)(1-2\lambda)\right]n - (1-\lambda)^2\right\}}, \tag{104}$$

then $\partial x^*/\partial\lambda \leq 0$ and $\partial p^*/\partial\lambda > 0$ (R_{I}). Simple calculations show that $\partial\underline{\beta}(\lambda)/\partial\mu > 0$. Since $d\mu/d\rho = -1/\rho^2 < 0$, we have that $\partial\underline{\beta}(\lambda)/\partial\rho < 0$. From (100) we obtain

$$\frac{\partial p^*}{\partial\lambda} = \frac{2\kappa}{(n-1)(1-\lambda)^2\left\{\frac{(n-1)(1-\lambda)\alpha S\tau}{n[\mu n+(n-1)(1-\lambda)]}\right\}^{\alpha}\tau}\vartheta_{\mathrm{CE}},$$

where

$$\vartheta_{\mathrm{CE}} \equiv (n-1)\left\{\left[\left(-\frac{\lambda^2}{2} + (\mu+1)\lambda - \frac{1+\mu}{2}\right)n + \frac{(1-\lambda)^2}{2}\right]\alpha + \frac{\lambda\mu n}{2}\right\}\beta + \frac{\mu n(1+\alpha)}{2}.$$

It follows that

$$\operatorname{sign}\left\{\frac{\partial p^*}{\partial\lambda}\right\} = \operatorname{sign}\left\{\vartheta_{\mathrm{CE}}\right\}.$$

Consequently,

$$\beta' = \frac{\mu n(1+\alpha)}{(n-1)\left(\left\{\left[\lambda^2 + (1+\mu)(1-2\lambda)\right]n - (1-\lambda)^2\right\}\alpha - \lambda\mu n\right)}. \tag{105}$$

Using that $\Gamma'' = 0$ and by replacing p^*, x^*, $\partial D_k(p^*)/\partial p_i$, $c'(Bx^*)$, $c''(Bx^*)$ and $\Gamma'(x^*)$ into (75) we obtain:

$$H = \frac{n\mu(1+\alpha)\tau B}{(n-\Lambda)\alpha\left[n(1+\mu) - \Lambda\right]}. \tag{106}$$

Note that $\operatorname{sign}\{\partial p^*/\partial\lambda\} = \operatorname{sign}\{H - \beta B\}$, so by solving $H - \beta B = 0$ for β we obtain again the expression for β' given by (105).

Recall that $CS'(\lambda) > (<)0$ iff $\beta > (<)\beta'$. The threshold β' is strictly increasing in λ:

$$\frac{\partial\beta'}{\partial\lambda} = \frac{\mu n(1+\alpha)}{(n-1)}\frac{\left[\mu n(1+2\alpha) + 2(1-\lambda)(n-1)\alpha\right]}{\left(\left\{\left[\lambda^2 + (1+\mu)(1-2\lambda)\right]n - (1-\lambda)^2\right\}\alpha - \lambda\mu n\right)^2} > 0.$$

As a result, $\lambda^o_{\mathrm{CS}} > 0$ if $\beta > \beta'(0)$, where

$$\beta'(0) = \frac{\mu n(1+\alpha)}{(n-1)\left[(1+\mu)n - 1\right]\alpha}.$$

Spillover thresholds and regions R_{I}, R_{II} and R_{III}[20]

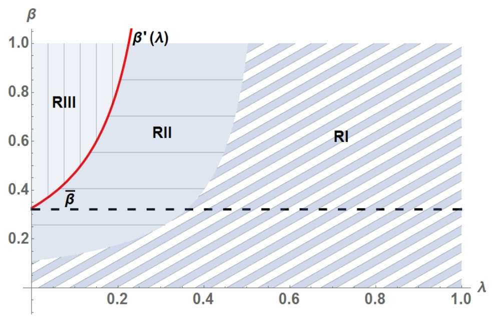

Fig. B11a. CE Bertrand model.

($n = 6,\ \alpha = 0.5,\ \rho = 0.5$)

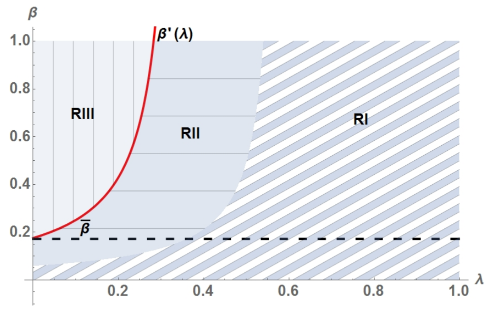

Fig. B11b. CE Bertrand model.

($n = 10,\ \alpha = 0.5,\ \rho = 0.5$)

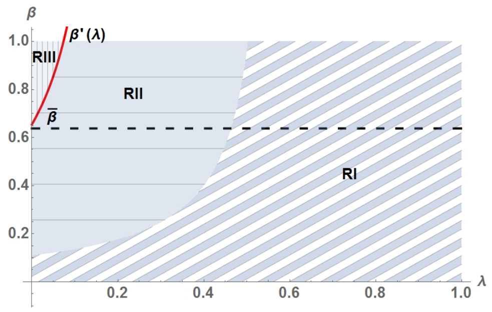

Fig. B11c. CE Bertrand model.

($n = 6,\ \alpha = 0.2,\ \rho = 0.5$)

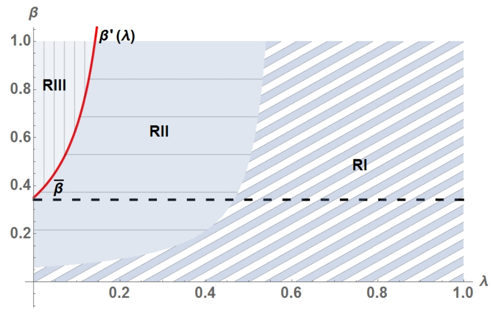

Fig. B11d. CE Bertrand model.

($n = 10,\ \alpha = 0.2,\ \rho = 0.5$)

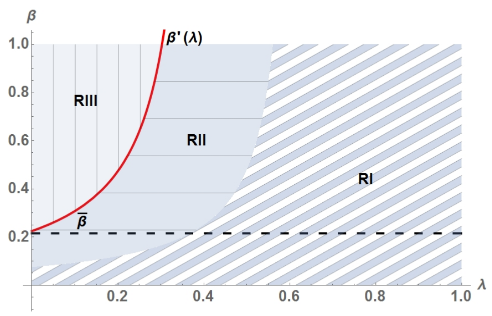

Fig. B11e. CE Bertrand model.

($n = 6,\ \alpha = 0.5,\ \rho = 2/3$)

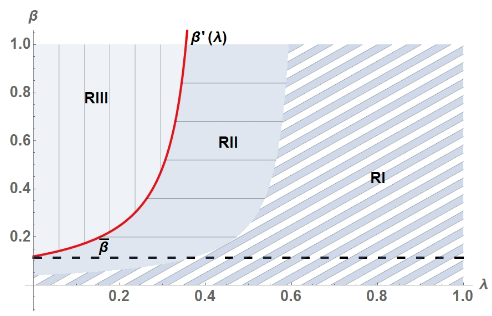

Fig. B11f. CE Bertrand model.

($n = 10,\ \alpha = 0.5,\ \rho = 2/3$)

We depict the spillover thresholds and the three regions in Figures B11a-f. For illustrative

[20] All simulations are conducted for $\kappa = 1$, $Y = 20$ and $\theta = 0.05$. Note that $S = Y/(1+\theta)$.

purposes, we consider six cases that differ in n, α and ρ. In contrast to the linear demand case, the condition under which only R_{I} exists for all β, which is given by (103), always holds for λ close or equal to 1. For lower values of λ, R_{II} and/or R_{III} may exist for β sufficiently high. Fig. 11a-d show how area R_{III} (respectively, R_{II}) increases (decreases) with α, and illustrate that R_{III} increases with n. Finally, the comparison of Fig. B11a with B11e, and B11b with B11f, display the increase of R_{III} with ρ.

Comparative statics on β'. Straightforward calculations show that $\beta'(\lambda)$ is strictly decreasing in α and strictly increasing in μ. Thus, $\partial\beta'(\lambda)/\partial\rho < 0$. As in the linear demand case, $\beta'(0)$ is strictly decreasing in n. Therefore, if $\beta'(0) > 1$ for $n = 2$, which holds when $\mu > \alpha/2$, then to have $\beta'(0) < 1$, so that $\lambda^{o}_{\mathrm{CS}} > 0$ when $\beta > \beta'(0)$, the number of firms must be sufficiently high such that

$$n > \frac{2(1+\mu)\alpha + \mu + 2\sqrt{\left[\left(\alpha+\frac{1}{2}\right)^2\mu + \alpha^2 + \alpha\right]\mu}}{2(1+\mu)\alpha}.$$

PROPOSITION BCE1 *Under the CE demand specification, if* $\mu n(2\Lambda - n) - (n-1)^2(1-\lambda)^2 > 0$ *then only region* R_{I} *exists. Otherwise, assume* $n > H(1)$, *where* H *is given by (106), and let* $\underline{\beta}(\lambda)$ *and* β' *be given, respectively, by (104) and (105). Then the following statements hold*:

(i) *if* $\beta \leq \underline{\beta}(\lambda)$, *then* $\frac{\partial q^*}{\partial\lambda} < 0$ *and* $\frac{\partial x^*}{\partial\lambda} \leq 0$ (R_{I});

(ii) *if* $\underline{\beta}(\lambda) < \beta \leq \beta'$, *then* $\frac{\partial q^*}{\partial\lambda} \leq 0$ *and* $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{II});

(iii) *if* $\beta > \beta'$, *then* $\frac{\partial q^*}{\partial\lambda} > 0$ *and* $\frac{\partial x^*}{\partial\lambda} > 0$ (R_{III}).

We have that $\underline{\beta}(\lambda)$ *and* $\beta'(\lambda)$ *are increasing in* λ *and decreasing in* ρ, *and* $\partial\beta'(0)/\partial n < 0$.

Profit. By inserting equilibrium values into the profit function and simplifying, we obtain:

$$\pi(\lambda) = \frac{1}{nB}\left[\frac{n\mu B - \alpha\tau(n-\Lambda)}{(\mu+1)n - \Lambda}\right]S.$$

Simulations show that also in Bertrand with CE demand, profit in equilibrium is strictly increasing in the degree of overlapping ownership: $\pi^{*\prime}(\lambda) > 0$.

Utility. Note that the indirect utility function in not linear in income. Thus, to solve the first-best problem we have to maximize the utility function subject to the resource constraint: $Y = \sum_{i=1}^{n} c_i q_i + \sum_{i=1}^{n}\Gamma(x_i) + q_0$. At the symmetric equilibrium the utility function with this constraint included is

$$V(\lambda) = n^{1/\rho}q^*\left(Y - nc(Bx^*)q^* - nx^*\right)^{\theta},$$

where $Y = S(1+\theta)$. Computing $V'(\lambda)$ and using the FOC $1 = -c'(Bx^*)\tau q^*$, after some manipulations we can write

$$V'(\lambda) = n^{1/\rho}\varrho^{\theta-1}\left\{[S(1+\theta) - nc(Bx^*)q^*(1+\theta) - nx^*]\frac{\partial q^*}{\partial\lambda} - c'(Bx^*)\beta(n-1)(1-\lambda)\theta n {q^*}^2\frac{\partial x^*}{\partial\lambda}\right\},$$

where $\varrho \equiv S(1+\theta) - nc(Bx^*)q^* - nx^*$.

We now may obtain the threshold $\bar{\beta}$ from the condition $W'(0) > 0$. In particular, the equation $W'(0) = 0$ is quadratic in β, and writes as $\vartheta_1\beta^2 + \vartheta_2\beta + \vartheta_3 = 0$, where

$$\vartheta_1 \equiv \alpha\left[(\mu+1)n-1\right](n-1)^2\left\{S\alpha\theta(n-1)^2Z^{-1} + \left[(\mu+1)n-1\right]n(1+\theta)n\mu\right\},$$

$$\begin{aligned}\vartheta_2 \equiv\ & -n(n-1)\left\{-\left[(\mu+1)n-1\right](1+\theta)\left[-(n-1)\alpha+n\mu\right](n-1) + S\mu\alpha^2\theta(n-1)^2Z^{-1}\right.\\ & \left.+\left[(\mu+1)n-1\right]^2\left[(n-1)\alpha^2-(1+\theta)(n-1)\alpha+n\mu(1+\theta)\right]\right\},\end{aligned}$$

and

$$\vartheta_3 \equiv -n^2\mu(\alpha+1)\left[(\mu+1)n-1\right]\left\{\left[(1+\theta)\mu-\alpha\right]n+\alpha\right\},$$

with

$$Z \equiv \frac{\alpha S(n-1)}{\left[(\mu+1)n-1\right]n}.$$

The threshold $\bar{\beta}$ is given by the positive root:

$$\bar{\beta} = \frac{-\vartheta_2 + \sqrt{\vartheta_2^2 - 4\vartheta_1\vartheta_3}}{2\vartheta_1}.$$

Table B5: H and Spillover Thresholds in CE Bertrand Model

$H =$	$n\mu(1+\alpha)\tau B/\left\{(n-\Lambda)\alpha\left[n(1+\mu)-\Lambda\right]\right\}$
$\underline{\beta}(\lambda) =$	$\mu n/\left((n-1)\left\{\left[\lambda^2+(1+\mu)(1-2\lambda)\right]n-(1-\lambda)^2\right\}\right)$
$\bar{\beta} =$	$\left(-\vartheta_2+\sqrt{\vartheta_2^2-4\vartheta_1\vartheta_3}\right)/(2\vartheta_1)$
$\beta' =$	$\mu n(1+\alpha)/\left[(n-1)\left(\left\{\left[\lambda^2+(1+\mu)(1-2\lambda)\right]n-(1-\lambda)^2\right\}\alpha-\lambda\mu n\right)\right]$

Threshold value $\bar{\beta}$

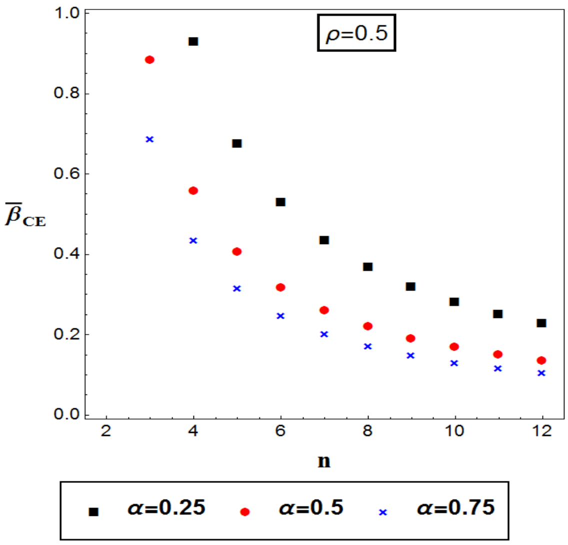

Fig. B12a.CE Bertrand model.

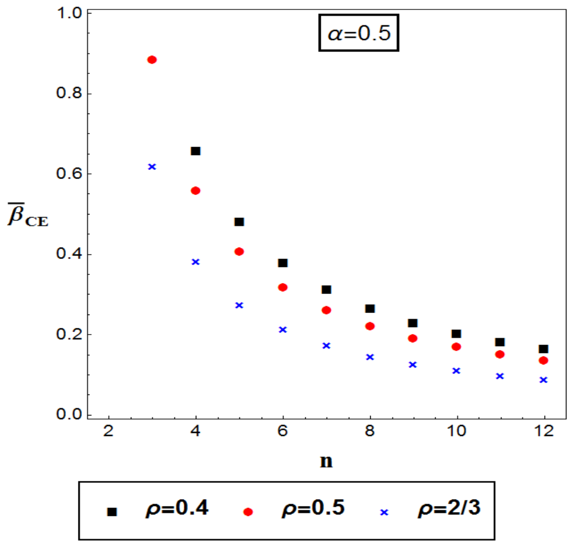

Fig. B12b. CE Bertrand model.

Comparative statics on $\bar{\beta}$. We observe in Fig. 12a,b that $\bar{\beta}$ decreases with α and ρ. The threshold as in the previous cases decreases with n and may take values greater than 1 (so $\lambda^o_{\mathrm{TS}} = 0$ irrespective of the value of β) when there are few firms in the market. Note that we use notation λ^o_{TS} with subscript TS even though we refer to utility V.

Comparative statics on the socially optimal degree of overlapping ownership. Simulation results are in line with previous findings: the socially optimal level of overlapping ownership increases with the size of spillovers (see Figures B13a-d) and with the number of firms (see Figures B14a-d).

Optimal degree of overlapping ownership (TS and CS standard)[21]

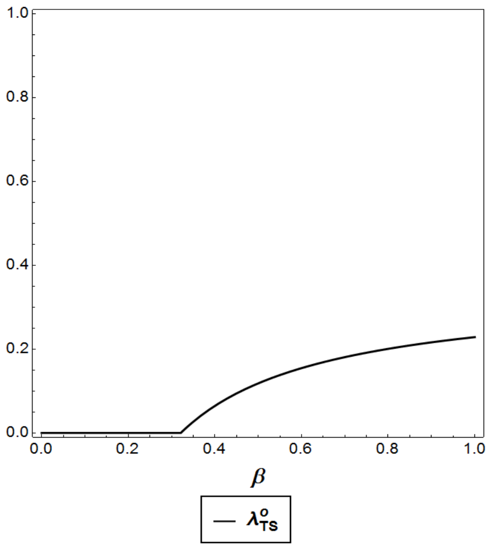

Fig. B13a. CE Bertrand model.

($\alpha = 0.5$, $\rho = 0.5$, $n = 6$)

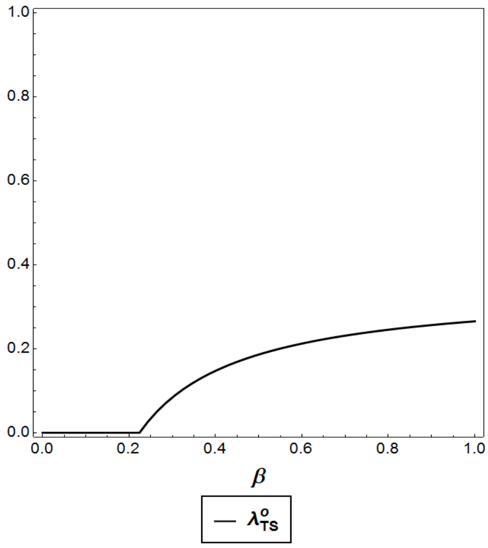

Fig. B13b. CE Bertrand model.

($\alpha = 0.5$, $\rho = 0.5$, $n = 8$)

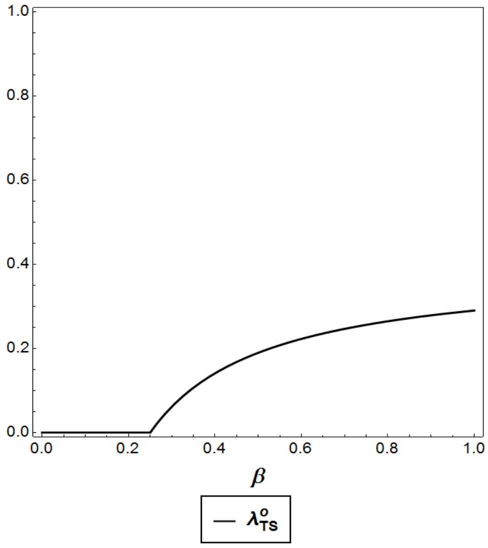

Fig. B13c. CE Bertrand model.

($\alpha = 0.75$, $\rho = 0.5$, $n = 6$)

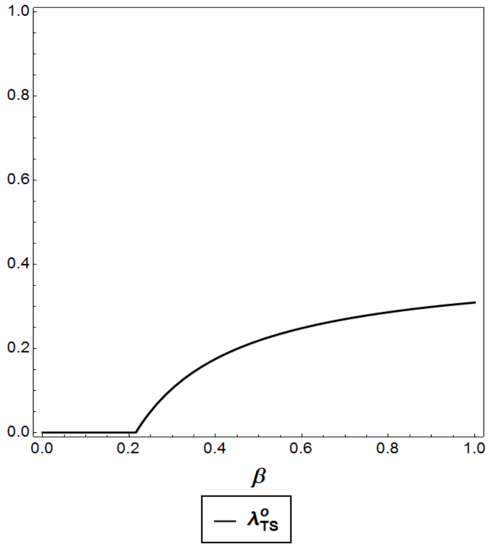

Fig. B13d. CE Bertrand model.

($\alpha = 0.5$, $\rho = 2/3$, $n = 6$)

[21] All simulations are conducted for $\kappa = 1$, $Y = 20$ and $\theta = 0.05$.

Optimal degree of overlapping ownership (TS and CS standard)[22]

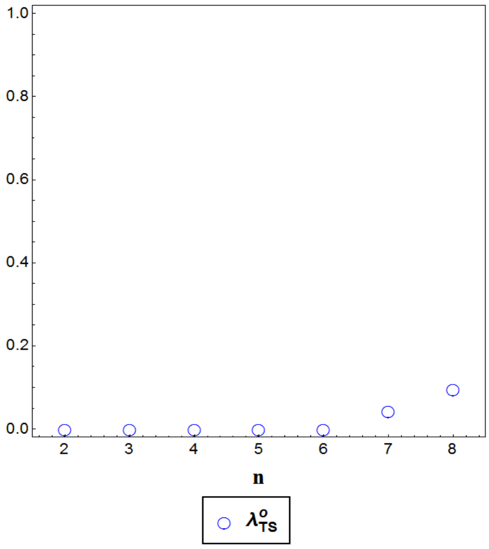

Fig. B14a. CE Bertrand model. ($\alpha = 0.5$, $\rho = 2/3$, $\beta = 0.2$.)

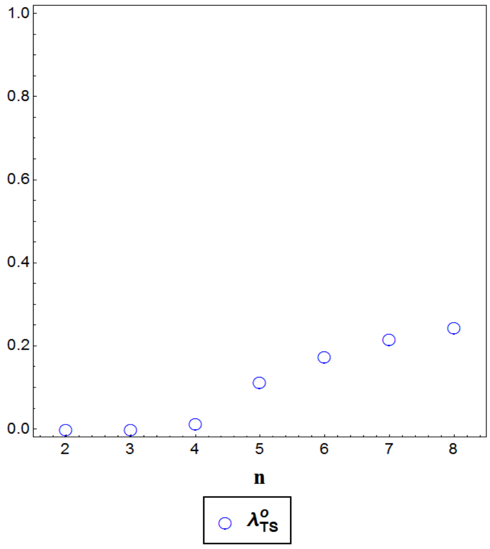

Fig. B14b.CE Bertrand model. ($\alpha = 0.5$, $\rho = 2/3$, $\beta = 0.4$.)

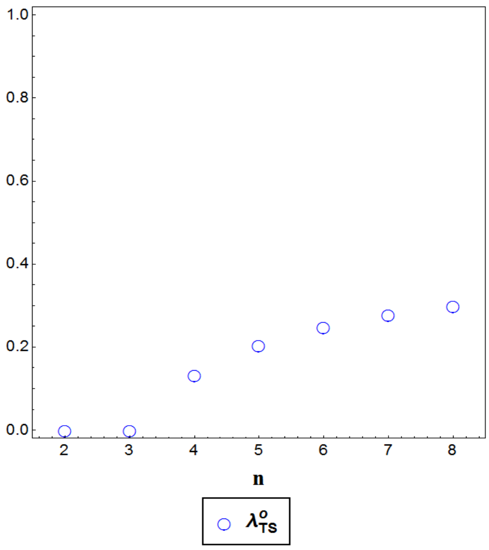

Fig. B14c. CE Bertrand model. ($\alpha = 0.5$, $\rho = 2/3$, $\beta = 0.6$.)

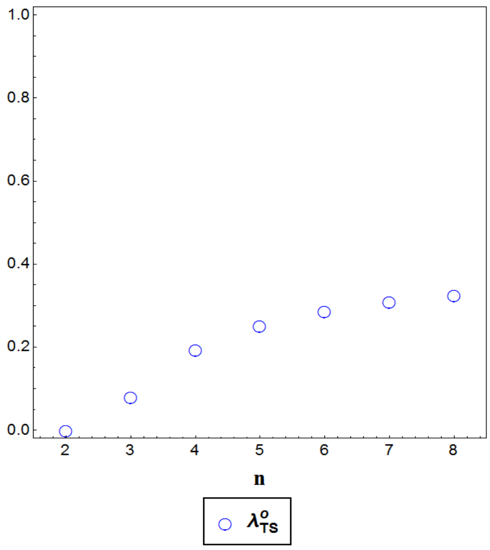

Fig. B14d. CE Bertrand model. ($\alpha = 0.5$, $\rho = 2/3$, $\beta = 0.8$.)

Comparative statics on the degree of product differentiation. In Fig. B15a-d we depict the optimal degree of overlapping ownership λ^{o}_{TS} for $\rho \in (0, 1)$; if $\rho \to 0^{+}$, then products tend to be independent, while if $\rho \to 1^{-}$, then products tend to be perfect substitutes. The grey area represents the values for ρ and λ where the interior (regular) equilibrium exists.[23] Simulations

[22] All simulations are conducted for $\kappa = 1$, $Y = 20$ and $\theta = 0.05$.

[23] That is, the second-order condition holds, and profit, cost, price, output and R&D are positive. (The regularity condition holds for $\lambda < 1$.)

show that for $\beta > 0$, λ^o_{TS} increases towards 1 when $\rho \to 1$. However, λ^o_{TS} is not U-shaped; the reason is that the monopoly case is not well defined with CE demand: when $\rho \to 0$, the price p tends to infinity, and therefore the output q tends to zero.

Optimal degree of overlapping ownership[24]

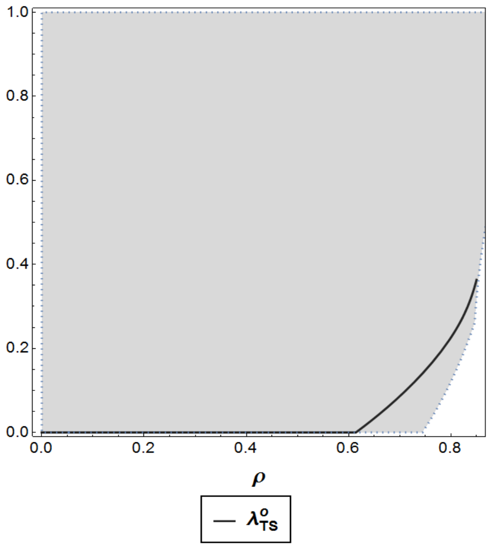

Fig. B15a. CE Bertrand model.

($\alpha = 0.5$, $\beta = 0.25$, $n = 6$)

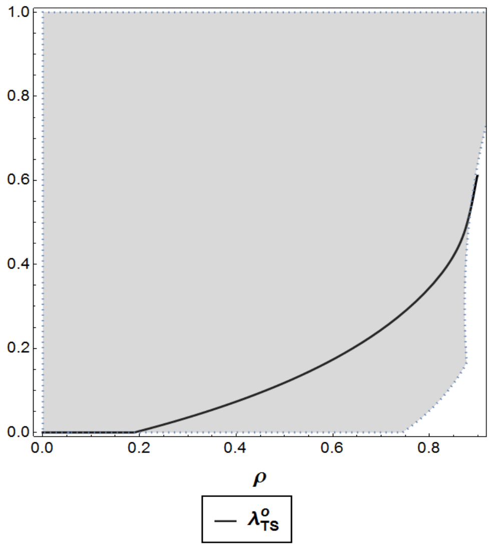

Fig. B15b. CE Bertrand model.

($\alpha = 0.5$, $\beta = 0.5$, $n = 6$)

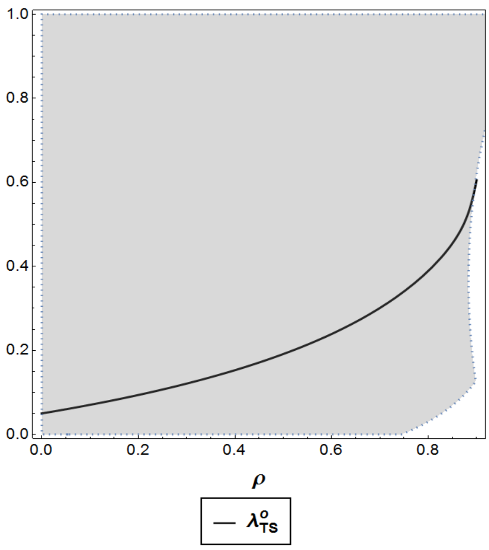

Fig. B15c. CE Bertrand model.

($\alpha = 0.5$, $\beta = 0.75$, $n = 6$)

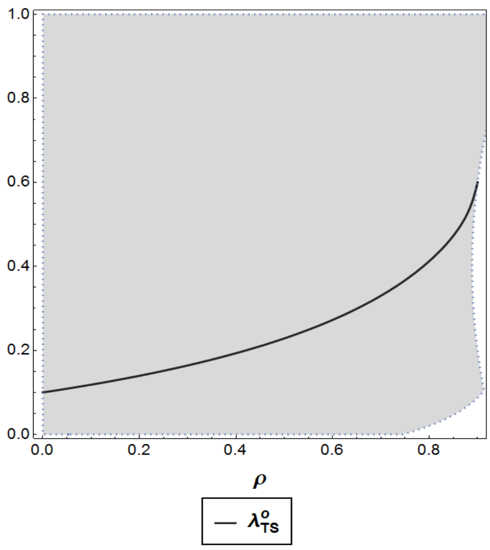

Fig. B15d. CE Bertrand model.

($\alpha = 0.5$, $\beta = 1$, $n = 6$)

[24] All simulations are conducted for $\kappa = 1$, $Y = 20$ and $\theta = 0.05$.

Two-stage model. Interior equilibrium. The interior equilibrium is characterized by the two FOCs (87) and (88), which at the symmetric equilibrium can be written as follows

$$q^* + (p^* - c(Bx^*))v_\lambda = 0$$

$$-c'(Bx^*)\tau q^* - 1 + \psi = 0.$$

Next we derive the strategic effect, $\psi \equiv (n-1)(\partial \phi_i / \partial p_j)(\partial p_j^* / \partial x_i)$.

Strategic effect. The expression for $\partial \phi_i / \partial p_j$, which is strictly positive for $\lambda < 1$, is given by (91). The expression for $\partial p_j^* / \partial x_i$ is computed in (82): $\partial p_j^* / \partial x_i = -\left(c'(Bx)/\Omega\right)(-\varphi)\left(\tilde{\beta}(\lambda) - \beta\right)$. By inserting equilibrium values into the definition of Ω, given in equation (83), we get

$$\Omega = \frac{\{(n-1)\left[(1+\mu)n + \lambda\right] + 1\}\left(\alpha \tau S A / n\right)^{4\alpha}(n-1)S^2 A^4 (1-\lambda)^2}{n^4\left[(n-1)(1-\lambda) + \mu n\right]\mu^2 \kappa^4} > 0.$$

The term φ is defined in (85). By replacing $\partial_{p_i p_i}\phi_i$, given by (102), Δ_p given by (101), and $\partial D_i(\mathbf{p}^*)/\partial p_i$ and $\partial D_k(\mathbf{p}^*)/\partial p_i$ provided in Table B3, into φ we obtain

$$\varphi = -\frac{A^4\left(\frac{\alpha \tau S A}{n}\right)^{4\alpha}(n-1)(1-\lambda)\left[n(1+\mu) + \lambda\right]S^2}{n^4 \kappa^4 \mu^2} < 0. \tag{107}$$

To obtain $\tilde{\beta}(\lambda)$ we first have to calculate $\partial_{p_i p_j}\phi_i$, which using equation (92) and Tables B3 and B4 can be shown to be

$$\partial_{p_i p_j}\phi_i = \frac{\Lambda(1-\lambda)S}{\left[(n-1)(1-\lambda) + n\mu\right]\mu p^2 n^2}.$$

As a result we have that

$$\frac{\partial D_i(\mathbf{p}^*)}{\partial p_i}\partial_{p_i p_j}\phi_i - \lambda \frac{\partial D_j(\mathbf{p}^*)}{\partial p_i}\partial_{p_i p_i}\phi_i = -\frac{A^4\left(\frac{\alpha \tau S A}{n}\right)^{4\alpha}(1-\lambda)S^2}{n^4 \kappa^4 \mu^2}, \tag{108}$$

which is strictly negative for $\lambda < 1$. By inserting (107) and (108) into (84), and simplifying, we get

$$\tilde{\beta}(\lambda) = -\frac{1}{(n-1)\left[(1+\mu)n + \lambda\right]} < 0.$$

Consequently, the strategic effect is:

$$\psi = -\frac{q^*}{v_\lambda}(n-1)(1-\lambda)\Lambda \frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}\left[-\frac{c'(Bx^*)}{\Omega}(-\varphi)(\tilde{\beta}(\lambda) - \beta)\right].$$

Let

$$\omega(\lambda) = \frac{\frac{\partial D_i(\mathbf{p}^*)}{\partial p_j}(n-1)(1-\lambda)\Lambda \varphi}{\Omega v_\lambda} > 0,$$

then the strategic effect is shown to be negative:

$$\psi = -c'(Bx^*)q^*\omega(\lambda)\left(\tilde{\beta}(\lambda) - \beta\right) < 0.$$